ILLINOIS UNIVERSITY
COMMITTEE ON SCHOOL
MATHEMATICS Unit 4

HIGH SCHOOL MATHEMATICS

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HIGH SCHOOL MATHEMATICS

Unit 4.

ORDERED PAIRS AND GRAPHS

UNIVERSITY OF ILLINOIS COMMITTEE ON SCHOOL MATHEMATICS

MAX BEBERMAN, *Director*

HERBERT E. VAUGHAN, *Editor*

Revised Edition

UNIVERSITY OF ILLINOIS PRESS · URBANA, 1962

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TEACHERS COMMENTARY

This unit is an introduction to the geometry of the number plane. It has proven to be one of the most popular of our units among both students and teachers. Its popularity among teachers may be attributed to the fact that we bring in a great variety of exercises. Teachers of conventional elementary algebra are accustomed to the fact that there is a very small number of types of graphing problems which students can be expected to do. Our problems on probability, the graphing of equations and inequations, both on the number plane and on the number plane lattice, and our work on operations with sets are welcome additions to the topic of graphing. Students enjoy this unit because they find frequent opportunities to use ingenuity in creating quick ways to solve what appear to be tedious problems, and because it helps them to visualize many properties of numbers.

You will be pleased to find that the students develop a high degree of skill in plotting points. We realize that it is important to develop in students the reflex of "go <u>over</u> or <u>back</u> with the first component, go <u>up</u> or <u>down</u> with the second component". To develop this reflex, students need to plot quite a few points. Students of moderate or high ability often resent [justifiably] such a routine kind of job. Your students will get much practice in plotting points, but they will enjoy this practice because it occurs in connection with interesting problems.

Students in conventional elementary algebra classes acquire the mental set that all graphs are straight lines. [A few of them may have graphed a parabola or two, or even an hyperbola.] The students who complete Unit 4 develop a much freer attitude. You can verify this by giving students in a conventional course the job of graphing the sentences;

$$xy = yx \qquad \text{and:} \qquad |x| + |y| = 3.$$

Conventionally trained students will try to put these sentences into some kind of "standard form"; they will probably assert that they can't make the graphs because they've never done problems like this before. UICSM students know what it means to graph a sentence. They know that they must find ordered pairs of numbers which satisfy the sentence, and then plot such ordered pairs until they see a pattern. Even the slowest students can make substitutions. The brightest ones will see a pattern almost immediately; the slowest ones may take considerably longer. But, ultimately, all students will solve the problem.

The work on factoring and exponents introduced in Unit 3 is considerably extended in Unit 4. Although this work is not an essential part of the work on graphing, it is an important aspect of the development of manipulative skills.

Parts A through E of the Miscellaneous Exercises contain important ideas and should be treated with care.

TABLE OF CONTENTS

[4-A]

s 25

A brief discussion of the two types of names is of interest. There is a case for calling a building 'the tool and die building' because if you know this name, you know something about what goes on in the building, even though you don't learn anything about its location from the name. But, when we abstract from buildings to points, the "tool and die" kind of name loses out entirely. The only property of a point in which we are interested is its location. Therefore, a name which tells you location is the obvious choice.

*

Ask your students what would be the disadvantage in numbering the buildings like this:

21, 5, 14, 9, 1

4, 8, 15, 6, 3

etc.

They will tell you immediately that it would be too hard to remember such a system. Since they have already learned the number names in a certain order, and since they can see an order in the arrangement of these 25 buildings, they will want these two orders to agree in some fashion. The students may ask in such a discussion why building 1 should not be the building in the upper left-hand corner. The only reasons for not choosing this building as building 1 [or the building in the upper right-hand corner or in the lower right-hand corner] are that we are heading toward a system which is similar to the one we have chosen, and that we are trying to establish the appropriate habits. They should understand that this method of assigning numbers to the buildings is an arbitrary one.

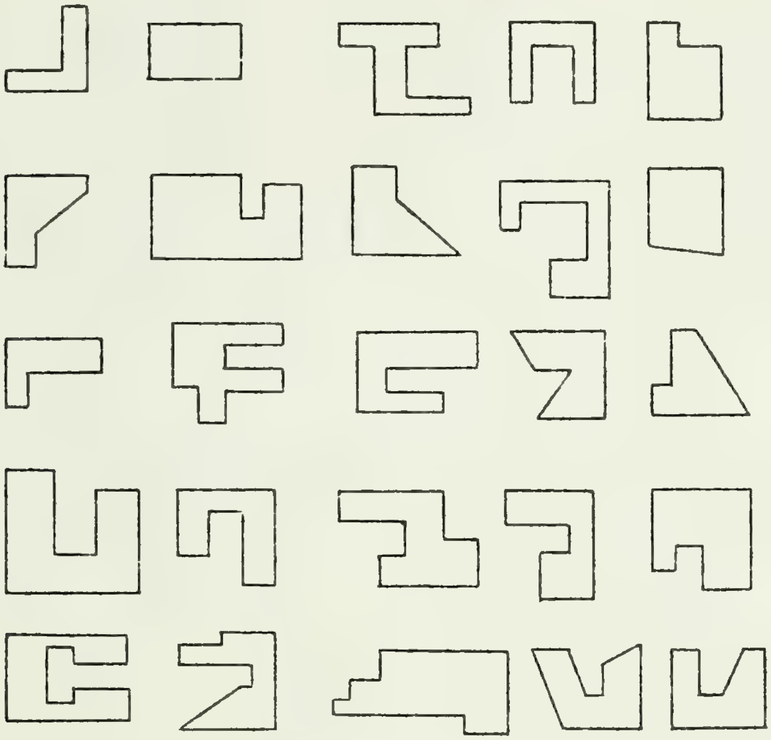
*

An interesting classroom exercise is to have students imagine that the diagram was extended "upward" indefinitely far, with 5 buildings to a row. Then, have students describe the location of buildings 26, 34, 47, 96, 105, 2007, and 1,000,000.

*

The assignment of numbers to rows and to columns as indicated at the bottom of page 4-B is, again, a matter of arbitrary choice. Point out to students the arbitrariness of this choice, and again indicate that we are trying to develop habits which will be in accord with the conventional method of numbering rows and columns.

Locating buildings.--Suppose that a big manufacturing plant has 25 buildings on its grounds. A map of the grounds looks like this:



Suppose you had to make up 25 names for these buildings so that a newcomer to the plant could find his way around as quickly and as easily as possible. Of course, you could just use 25 names like 'tool and die building' or 'administration building' or 'spare parts warehouse'. Such names would tell a newcomer something about what went on in the buildings, but they would not help him learn where the buildings were located.

Names which are easy to remember and which could be used in locating the buildings are the numerals for whole numbers from 1 through 25. These numerals are already known (to most people) and, if they were used to name the buildings in some order, say, like this:

21	22	23	24	25
16	17	18	19	20
11	12	13	14	15
6	7	8	9	10
1	2	3	4	5

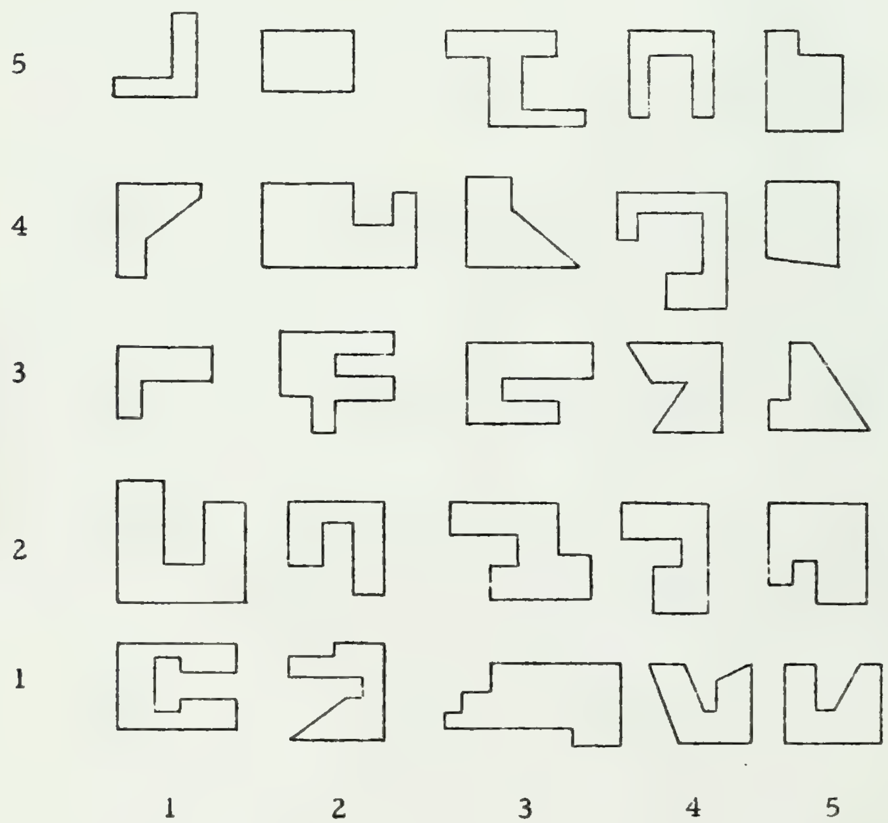
it would be easy to learn where the buildings were located. If you told a newcomer to go to building 14, he would think:

> Building 14; there are 5 buildings in each row, so the first row is 1 to 5, the second row is 6 to 10, and the third row is 11 to 15. So, building 14 is the fourth building in the third row.

Notice that to tell himself the location of building 14 he used a single number, 14, to get a pair of numbers:

building 4 in row 3.

Since you think of the building in terms of "which building in which row", you might just as well have named them that way in the first place. Say, like this:



[4-C]

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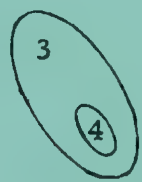
The method chosen for naming an ordered pair is probably one which is most convenient for printing purposes. Obviously, there could be other methods of naming ordered pairs. You might want to have students propose other methods. For example, for the ordered pair (4, 3), they might use:

$\boxed{4 \;|\; 3}$

or: 4 → 3

or: $\begin{array}{|c|}\hline 4 \\ \hline 3 \\ \hline\end{array}$

or:



Point out that in each of these methods of naming an ordered pair, there must be a convention which tells which component is first and which component is second.

*

Since an ordered pair is not the set consisting of its components [in particular, $(4, 3) \neq \{4, 3\}$], it is improper, as well as confusing, to speak of the components of an ordered pair as its first and second members. An ordered pair can be defined to be a set of a certain kind, but the members of this set will not be the components of the ordered pair. [According to one common definition, $(4, 3) = \{\{4\}, \{3, 4\}\}$. So, the members of (4, 3) are $\{4\}$ and $\{3, 4\}$, and neither member of (4, 3) is a component of (4, 3).]

Instead of 'building 14', you will now say: building 4 in row 3. Instead of assigning a single number to each building, you are describing the location of each building in terms of two numbers, its "column number" and its "row number". But just the two numbers themselves are not enough. To use them to locate a building, you must know which of the two numbers is its column number [and which is its row number]. So, it would not be correct to say that you are now locating each building merely by assigning a set of numbers to it [instead of, as at first, a single number]. To direct a newcomer to a particular building, you must say:

Go to building 4 in row 3.

But, for those in the plant who are used to this numbering system, you would probably get in the habit of saying:

Go to building 4, 3

or even:

Go to 4, 3.

You would expect that such people understood your convention that the number you mention first is the column number of a building and the number you mention second is its row number. What this amounts to is that you are now assigning to each building an ordered pair of numbers. In describing the location of a building [in the case of the instructions above] you do not use the set whose members are the numbers 3 and 4, but the ordered pair whose first component is 4 and whose second component is 3. It is customary to name this ordered pair '(4, 3)'. Although $\{4, 3\} = \{3, 4\}$,

$$(4, 3) \neq (3, 4)$$

because the first component of (4, 3) is not the first component of (3, 4). [Also, $(4, 3) \neq (4, 2)$ because (4, 3) and (4, 2) have different numbers as second components.] In general,

$$\forall_a \forall_b \forall_c \forall_d \; [(a, b) = (c, d) \text{ if and only if } (a = c \text{ and } b = d)].$$

EXERCISES

A. Arrange 16 dots on a sheet of paper in a four-by-four square array. Label the columns '1', '2', '3', and '4' from left to right; label the rows '1', '2', '3', and '4' from bottom to top. Agree that the first component of an ordered pair of numbers will tell you the column, and the second component will tell you the row. Find the dots which correspond with these ordered pairs. [Darken the corresponding dot and write the name of the ordered pair next to it.]

1. (3, 2)	2. (2, 3)	3. (4, 1)
4. (3, 3)	5. (1, 2)	6. (2, 1)
7. (8 - 6, 3)	8. (4, 10 - 6)	9. $(2 \div 2,\ 15 \div (5 \times 1))$

B. The set of numbers consisting of 3 and 5 is the same as the set consisting of 5 and 3. That is,

$$\{3, 5\} = \{5, 3\}.$$

But,

$$(3, 5) \neq (5, 3).$$

Explain.

C. Consider the sets A and B where

$$A = \{3, 5, 8, 9\}$$

and $$B = \{4, 5, 6, 7, 8\}.$$

We can build ordered pairs from these sets by selecting first components from one of the sets and second components from one of the sets.

Sample. List all the ordered pairs whose first components belong to A and whose second components belong to B.

Solution. [See the list at the top of the next page.]

Part A should be done as an "in class" assignment. Move among the students and observe their facility in doing the work.

*

Answer for Part B.

(3, 5) and (5, 3) are different ordered pairs because they have different first components [also, because they have different second components].

{3, 5} is the same set as {5, 3}; this set is the set consisting of just 3 and 5.

*

Answers for Part C [on pages 4-D, 4-E, 4-F].

1. (4, 3) (4, 5) (4, 8) (4, 9) (5, 3) (5, 5)
   (5, 8) (5, 9) (6, 3) (6, 5) (6, 8) (6, 9)
   (7, 3) (7, 5) (7, 8) (7, 9) (8, 3) (8, 5)
   (8, 8) (8, 9)

2. (3, 3) (3, 5) (3, 8) (3, 9) (5, 5) (5, 8)
   (5, 9) (5, 3) (8, 8) (8, 9) (8, 3) (8, 5)
   (9, 9) (9, 3) (9, 5) (9, 8)

3. 35

*

Notice [on page 4-E] the heavy multiplication sign used in naming cartesian products. Notice, also, that the word 'multiplication' is receiving another extension in meaning. This time the word refers to an operation with sets. Moreover, this is an example of an operation which is not commutative. [Exercise 3 of Part C suggests the reason for calling this operation on sets 'multiplication'.]

Exercises 4, 5, 6, and 7 on page 4-F provide practice in combinatorial problems, a kind of problem which occurs frequently in studying probability.

[4-D]

A.

(3, 4)	(5, 4)	(8, 4)	(9, 4)
(3, 5)	(5, 5)	(8, 5)	(9, 5)
(3, 6)	(5, 6)	(8, 6)	(9, 6)
(3, 7)	(5, 7)	(8, 7)	(9, 7)
(3, 8)	(5, 8)	(8, 8)	(9, 8)

1. List all the ordered pairs whose first components are elements of B and whose second components belong to A.

2. List all the ordered pairs each of whose components is a member of A.

3. Suppose M is a set consisting of 5 numbers and N is a set consisting of 7 numbers. What is the total number of ordered pairs whose first components belong to M and whose second components belong to N?

*

The set of all ordered pairs with first components from A and second components from B is called the cartesian product of A by B, and is named by the symbol:

$$A \times B \qquad \text{[read as 'A cross B'].}$$

For example, if A = $\{1, 3, 4\}$ and B = $\{1, 2\}$, then

$$A \times B = \{(1, 1), (1, 2), (3, 1), (3, 2), (4, 1), (4, 2)\}$$

and

$$B \times A = \{(1, 1), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4)\}.$$

Here are other examples of cartesian products.

$$\{5, 7\} \times \{8, 1\} = \{(5, 8), (5, 1), (7, 8), (7, 1)\}$$

$$\{9\} \times \{3, 8, 9\} = \{(9, 3), (9, 8), (9, 9)\}$$

['cartesian' is derived from the Latinized form of the name of Rene Descartes (dā - kärt'), a French mathematician and philosopher who lived in the first half of the seventeenth century.]

*

4. Describe each cartesian product by listing its elements (as on page E).

(a) $\{4, 1, 7\} \times \{8, 5\}$ (b) $\{3, 1\} \times \{9, 1, 3\}$

(c) $\{8, 5\} \times \{4, 1, 7\}$ (d) $\{5, 7\} \times \{5, 7\}$

(e) $\emptyset \times \{2, 8\}$ (f) $\{2, 8\} \times \emptyset$

5. If M is a set of m numbers and N is a set of n numbers, $M \times N$ consists of __________ ordered pairs of numbers and $N \times M$ consists of __________ ordered pairs of numbers.

6. Describe a set S and a set T for which $S \times T = T \times S$.

7. A baseball squad has 6 pitchers and 3 catchers. How many possible batteries are there? [A battery is a pair consisting of a pitcher and a catcher.]

8. If you throw two dice, you get a pair of numbers. For example, if you throw this:

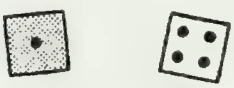
you get the pair $\{4, 6\}$. This is not an ordered pair, and, usually, you don't care because you add the numbers anyway. [Explain.]

You could use the dice to obtain an ordered pair, by agreeing, for example, that the die which lands first gives the first component of the ordered pair.

An easier way to get ordered pairs of numbers from the dice is to have one die red and one white, and agree that the red die gives the first component of the ordered pair. In the picture below, assume that the shaded die is the red one. Tell, in each case, the ordered pair given by the dice.

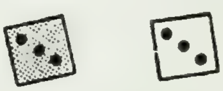
(a) 

(b) 

(c) 

(d) 

(e) 

(f) 

(g) How many ordered pairs of numbers could you get from two such dice?

4. (a) $\{(4, 8), (4, 5), (1, 8), (1, 5), (7, 8), (7, 5)\}$

   (b) $\{(3, 9), (3, 1), (3, 3), (1, 9), (1, 1), (1, 3)\}$

   (c) $\{(8, 4), (8, 1), (8, 7), (5, 4), (5, 1), (5, 7)\}$

   (d) $\{(5, 5), (5, 7), (7, 5), (7, 7)\}$

   (e) There are no elements in this cartesian product. [The cartesian product is $\emptyset$.]

   (f) There are no elements in this cartesian product. [The cartesian product is $\emptyset$.]

   [In both (e) and (f), one of the sets named is $\emptyset$. Since the empty set has no elements, it is impossible to form ordered pairs using elements of $\emptyset$ as components.]

5. mn, nm

6. [There are many such descriptions. We give three here; your students will doubtless suggest others.]

   (a) If $S = \{2, 3\}$ and $T = \{2, 3\}$ then $S \times T = T \times S$.

   (b) If $S = \emptyset$ and $T = \{5, 7, 11\}$ then $S \times T = T \times S$ [$= \emptyset$]

   (c) If $S = \{8, 5, 1\}$ and $T = \emptyset$ then $S \times T = T \times S$ [$= \emptyset$]

   [In general, for each set S and each set T, $S \times T = T \times S$ if and only if $S = \emptyset$ or $T = \emptyset$ or $S = T$.]

7. 18

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Exercise 8, and Part D which begins on page 4-G, involve important ideas. Exercise 8 illustrates in a striking manner the distinction between an ordered pair of numbers and a set of two numbers. If you say the names of two numbers, you must say them in order with respect to time. Therefore, there is a tendency to take the numbers as an ordered pair, and the only way to prevent this is to declare that you didn't mean to assign an order. Similarly, when you write the names for the numbers, unless you write them on separate sheets of paper, there is a tendency to interpret them as being given in order since our conventional manner of writing involves a left-to-right order. So, we need to give the student a good example of an occasion where one is concerned with unordered pairs of numbers. Such an occasion is one

[4-F]

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in which a student throws two indistinguishable dice. From the dice, he obtains two numbers. Which one is the first number? This is a silly question for you don't have an ordered pair. [And in games in which a student does throw a pair of dice, he is interested in the sum. Since addition is commutative, the notion of order is unimportant.] But, if you make one die red, or mark it in some other way, and agree that from it you will obtain the first number, then throwing a pair of such dice will give you an ordered pair.

Part D asks the students to carry out an experiment in dice-throwing with two differently colored dice. We expect this experiment to give students some ideas of the elementary theory of probability, to enable them to acquire skill in plotting ordered pairs, and to develop in them some intuition concerning equally likely events. This intuition will be used in answering the questions of Part E.

*

We have been asked, usually in jest, if dice-throwing constitutes appropriate subject matter for high school students. If you should be challenged on this question, you might point out that dice are used in a wide variety of quite innocent games played by children. Moreover, we are using dice as convenient equipment for a little bit of experimental "research" in probability theory; what other people may do with dice is their problem! However, in obtaining dice for Parts D and E on page 4-G, the school or the teacher should purchase them from a local toy or sporting goods store rather than expect students to provide them. The dice should be kept in school as "laboratory" equipment, and not "loaned out" for homework.

*

8. (a) (5, 2) (b) (4, 1) (c) (1, 4) (d) (4, 1) (e) (3, 3)
   (f) (3, 3) (g) 36

4.

[4-G]

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is a perfectly good way to talk about odds, but it is not easily adapted to probability. You will have to ask such a student to use the other method so that the entire class will talk the same language.

*

Answers and comments for Part E.

In Exercise 1, the student should say something equivalent to "the chances are 1 in 6" as the answer for all 5 questions.

In Exercise 2, he should give an answer equivalent to "1 in 36" for each of the 6 questions.

In Exercise 3, the student should reason somewhat along these lines:

> The chances of getting the ordered pair (3, 2) by itself are 1 in 36. The chances of getting (1, 5) by itself are 1 in 36. Then [intuition tells me], the chances of getting either (3, 2) or (1, 5) are 2 in 36, which is the same as 1 in 18.

Don't give the student any rule here; let him figure it out himself. He should quickly come to see that the chances of getting 1 of 2 points are 1 in 18. He probably has in the back of his mind the idea that if he were asked the chances of landing in any given subset of these 36 points he would count the number of points in the subset and then say that the chances were that number in 36. He can test this presently unverbalized rule on the last question in Exercise 3 because he has already agreed in Exercise 1 that the chances of getting into a given row [or a given column] are 1 in 6.

In Exercise 4, the student should be able to formulate by himself the correct rule:

> Count the points in the set in question. The chances are this number in 36.

By now, students should be feeling the need for a more convenient language.

[4-G]

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came out five heads and five tails.] Even if you say "in the long run the number of heads and the number of tails will get closer and closer together" you will probably be wrong because if you actually make tosses it is highly likely that the difference between the number of heads and the number of tails will increase in the long run. [Notice how vague expressions like 'highly likely' and 'in the long run' creep into the discussion.] Now if you said that the quotient of the number of heads by the number of tails will get "closer and closer" to 1, you would be getting warm, but you would still be wrong. It is quite possible that after 100 tosses the ratio is, say, 0.97, and after 200 tosses the ratio is, say, 0.95. Again, to talk about events such as these, you would go back to discussing their probability. Well, one can go on and on setting up straw dummies and knocking them over like this, but we have gone far enough. You can say that the ratio of heads to tails will, in some sense, approach 1. But to make accurate the phrase 'in some sense' takes a great deal of technical machinery. One thing that this does not mean is that if you have been tossing coins for a while and the heads have a large lead on the tails, the chances that you will get a tail are greater than before. Students often advance such an erroneous theory saying that it follows from "the law of averages". If a student believes this, carry out the following experiment. Toss a coin in groups of three tosses. If the outcome of a group of three tosses is anything other than three heads, ignore it [you'll be ignoring about seven eighths of your groups of three]. But if the outcome is three heads, make another single toss and record its outcome. Do this many times. It will be apparent that tossing a coin after three heads in a row have been tossed does not bias the outcome of the toss.

By now you may be thinking that if probability is as tricky as we claim, why teach it at all. It isn't tricky or difficult as long as 'probability' is left as a word without further definition or interpretation, but which one comes to understand by experience and intuition. You all know the wide applications of probability to nearly all kinds of organized research; this is one reason for teaching it. Another is that it is interesting to students at this age. [Two good references are: Kemeny, Snell, and Thompson, Introduction to Finite Mathematics (New York: Prentice-Hall, 1957), and: Commission on Mathematics, Introductory Probability and Statistical Inference for Secondary Schools (New York: College Entrance Examination Board, 1957).]

One other detail. You may find a student who prefers to say that the odds, or chances of getting, say, 3, when throwing one die are 1 to 5. He is considering the 6 equally likely events and saying that 1 of them gives the outcome 3 and that 5 of them do not give this outcome. This

[4-G]

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After throwing the dice, the student will feel intuitively that the chances of any one face of a die being uppermost are the same as the chances for any other face. Some students may already be able to express themselves quantitatively. For example, they may say "the chances are 1 out of 6", or "there is 1 chance in 6". The student should believe because of the general nature of the results of many throws and, perhaps just as importantly, because of his intuitive feelings about the symmetry of a die, that each of the 6 possible outcomes of a throw is equally likely. Here, 'equally likely' is a primitive term. We can define the probability of any one of these equally likely events to be the reciprocal of the number of events. Thus, we are choosing a measuring scale where probability 1 is associated with an event which is "certain" to occur [such as getting 1 or 2 or 3 or 4 or 5 or 6 from throwing a die], and where probability 0 is associated with an event which is not among the possible outcomes of the experiment [such as getting 62 as the sum when throwing two dice]. In elementary probability problems of the type the student will be working, the real problem is to get a clear picture of what are the underlying equally likely events to which probabilities can easily be assigned. The physical experiment is to make reasonable for the students the assigning in Part E of probability 1/6 to each of the events: getting 1, getting 2, getting 3, getting 4, getting 5, and getting 6.

*

Before going into the details of the exercises of Part E, we insert here a note of caution concerning the treatment of probability. Be very careful in this work to avoid trying to say what 'probability' means. Keep the discussions centered around the simple questions being asked and the finding of probabilities rather than their interpretation. For example, if you flip a balanced coin, it is usually assumed that "getting heads" and "getting tails" are equally likely events, and therefore that the probability of either of these events is 1/2. Now, what does this last sentence tell you? If you answer, "the odds are even", or "there's as good a chance of getting one as there is the other", or "you are just as likely to get one as the other", you are safe because you have said no more than what was intended by the sentence. But if you say: "you'll get just as many heads as you will tails", you are wrong. The probability of getting exactly 5 heads in 10 tosses of a coin is small. [If you should run into someone who doesn't believe this, have him make tosses with a coin, ten tosses at a time, and record the total number of ten tosses made and also the number of ten tosses which

[4-G]

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[The first year that the exercise in Part D was included in our text we asked the teachers in our pilot schools to send us the results obtained by their classes. These were compiled, and the table below was prepared. We include it as a matter of interest for you; you may want to give your class some of the data.]

<u>Results of the UICSM Experiment With Dice</u>

SECOND NUMBER \ FIRST NUMBER	1	2	3	4	5	6	Row Totals	Percent of Total in Each Row
Percent of Total in Each Column	16.8	16.1	16.4	15.6	16.9	18.2		
Column Totals	1590	1526	1551	1480	1605	1723	Grand Total 9475	
6	251	274	267	233	255	280	1560	16.5
5	236	293	270	248	283	291	1621	17.1
4	381	246	278	254	242	315	1716	18.1
3	240	240	230	254	261	267	1492	15.7
2	235	242	239	253	270	268	1507	15.9
1	247	231	267	238	294	302	1579	16.7

[The expected frequency for each ordered pair for 9475 throws is 263.]

D. Obtain a pair of dice with one die red and one die white. [Any two colors will do, but if you don't use red and white, you will have to interpret the instructions accordingly.] Make 36 dots in a 6-by-6 square array, placing the dots about one inch apart. Assign an ordered pair of numbers to each of these dots, following the convention that the first component in the ordered pair tells you the column the dot is in, and the second component tells you the row the dot is in. The columns are to be numbered from 1 to 6 starting at the left, and the rows are to be numbered from 1 to 6 starting at the bottom. So, the first and second components of the 36 ordered pairs are chosen from {1, 2, 3, 4, 5, 6}. Now, if we agree that the red die gives us the first component of an ordered pair then each throw of this pair of dice gives us one of these 36 ordered pairs of numbers.

Throw the dice. Make a small tally mark next to the dot which corresponds with the ordered pair given by the dice. Repeat this process several hundred times. [This could be a class project with each student making and recording, say, 25 throws and combining all the results.]

E. Refer to a 6-by-6 chart like the one in Part D in answering these questions.

1. What are the chances that you will get a dot in the third row when you make a throw? First row? Fourth row? First column? Third column?

2. What are the chances that you will get the dot corresponding to (2, 3) when you make a throw? (4, 2)? (1, 5)? (2, 4)? (3, 3)? Any particular dot?

3. What are the chances of getting in one throw either (3, 2) or (1, 5)? Either (1, 6) or (6, 1)? Either (1, 2) or (1, 3) or (1, 4)? Either (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), or (2, 6)?

4. Pick out any subset of this set of 36 dots. Describe a method for telling the chances that you will get a dot in this subset in one throw.

(continued on next page)

5. Next to each of the dots in your diagram write a numeral for the sum of the components in the ordered pair corresponding with this dot. How many dots are there for which the sum is 12? What are the chances of getting 12 when you throw a pair of dice? How many dots are there for which the sum is 7? What are the chances of getting 7 when you throw two dice?

   Make a table showing the chances of getting each of all possible sums for the two dice. Indicate the chances by giving the <u>probability</u> of getting each sum. You compute the probability like this. If the chances are, say, 6 in 36, then the probability is the number $\frac{6}{36}$. Add up all the probabilities listed in your table for the 11 sums.

   What is the probability of getting any one of the sums 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, or 12 when you throw two dice? What is the probability of getting the sum 89 when you throw two dice?

6. Find the subset of dots where the sum is either 6, 7, or 8. What is the probability of getting in this subset when you throw two dice? What is the probability of getting any one of the sums 6, 7, or 8 when you throw two dice? What is the probability of getting either the sum 3 or the sum 10 when you throw two dice? What is the probability of getting both the sums 3 and 10 in one throw of the dice?

7. (a) What is the probability of getting a sum which is an even number? An odd number? An even number or an odd number?

   (b) What is the probability of getting an ordered pair with both components even numbers? With both components odd numbers? Is the sum of these probabilities the same as the probability of getting a sum which is an even number?

8. If you were to throw three dice at once, what is the probability that you would get the sum 3? 18? 5?

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Exercise 5 provides an opportunity for introducing the word 'probability'. The student has had experience in the earlier exercises in "combining chances", so he is ready for the notion that chance can be measured by numbers. And, in fact, the probability of an event is a number between and including 0 and 1.

Students should have no trouble with Exercise 5 providing they actually carry out the instructions. They will find that there is only 1 point with a '12' beside it, a corner point. The probability of getting this point is 1/36, and this gives the probability of getting 12 as the sum. The student will find that there are 6 points with a '7' written beside each. The probability of getting 7 as the sum is 6/36, or 1/6. You may have to give a few more examples of how probabilities are found but your students will probably accept this without difficulty. Near the end of Exercise 5, we ask the student for the probability of "some outcome". He will probably give '1' as a response without even using his rule, but he should see that he can obtain this answer by formal means, also. He counts up all the points indicated and gets 36. Then he divides by 36, as he has been doing to find probabilities, and gets 1. Similarly, he will laugh at the idea of getting 89 when throwing 2 dice. But he should see that of his 36 possible outcomes, 0 of them give him the sum 89, and 0/36 = 0.

In Exercise 6, the subset of dots with sum either 6, 7, or 8 contains 16 dots. Thus, the probability of getting in this subset--that is, of getting either 6, 7, or 8--is 16/36, or 4/9. The students should recognize that the first two questions of Exercise 6 are equivalent. Since there are 2 dots for which the sum is 3, and 3 dots for which the sum is 10, the probability of getting either the sum 3 or the sum 10 is 5/36. The probability of getting both the sums 3 and 10 in 1 throw of the dice is 0, since this event is impossible.

In Exercise 7(a), the probability of getting a sum which is an even number is computed in the following manner. Possible even sums are 2, 4, 6, 8, 10, and 12. The number of points corresponding with each of these is 1, 3, 5, 5, 3, and 1, respectively. So, there are 18 dots which correspond with even sums. Hence, the probability of getting a sum which is an even number is 18/36, or 1/2. Since a sum is either an even number or an odd number, and since there are 18 even sums and 36 sums altogether, it follows that the number of ways of getting an odd sum is 18. Therefore, the probability of getting a sum which is an odd number is 18/36, or 1/2. The probability of getting an even sum or an odd sum is 36/36, or 1. This event is certain to happen.

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In considering 7(b), we know that the ordered pairs with both components even numbers are the ordered pairs in the cartesian product of $\{2, 4, 6\}$ by itself. There are 9 such ordered pairs. [Similarly, there are 9 ordered pairs $[\{1, 3, 5\} \times \{1, 3, 5\}]$ whose components are odd numbers.] Hence, the probability of getting an ordered pair with both components even numbers is 9/36 [or: 1/4]. [Similarly, the probability of getting an ordered pair with both components odd numbers is 1/4.]

There is another way in which you can determine the number of ordered pairs whose components are both odd numbers. Note that the only ways in which an even number sum can be obtained is to add an even number to an even number or to add an odd number to an odd number. Since there are 18 even number sums altogether, and 9 of these are obtained by adding an even number to an even number, it must be the case that the other 9 are obtained by adding an odd number to an odd number.

The sum of the two probabilities is, of course, the same as the probability of getting a sum which is an even number.

Exercise 8 is a test of the students' ability to generalize. Students will probably get solutions by a variety of correct methods. Here is the method we would like the class ultimately to settle on. If the three dice were colored, a throw would give an ordered triple of numbers. There are $6 \times 6 \times 6$, or 216 such ordered triples, and they are all equally likely events. Graphically, think of a cubical array of dots with six dots in each edge. A probability of 1/216 is assigned to each of these 216 points. We are now ready to answer questions. First question: What is the probability of getting the sum 3? Answer: There is only one point in the cube which gives the sum 3. It is the corner point (1, 1, 1). Since there is only one such point, the probability of getting the sum 3 is 1/216. Second question: What is the probability of getting the sum 18? Answer: Again, there is only one point which gives the sum 18. It is the corner point [diagonally across from the point considered previously] (6, 6, 6). Since there is only one point, the probability is again 1/216. In the third question, the student should realize that the whole thing amounts to finding how many points in the cube array give sum 5 and that this amounts to finding how many ordered triples there are the sum of whose components is 5. By trial, he finds

(1, 1, 3),	(2, 1, 2),		
(1, 2, 2),	(2, 2, 1),		
(1, 3, 1),		and	(3, 1, 1).

He organizes his work by first looking for triples with first number 1,

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and then looking for triples with first number 2, etc. Since, there are 6 points, the probability of getting the sum 5 is 6/216, or 1/36. Again the student should see that he has all the machinery he needs to tell the probability of obtaining any sum, but the job of counting points would become laborious for the sum, say, 9. Urge him to find short cuts for counting the points in sets like these.

*

Here are other questions you might use in considering the outcomes of a throw of two differently colored dice.

(1) What is the probability of getting an ordered pair with components whose

   (a) sum is 5? [Answer: 1/9]

   (b) sum is 7 or 11? [Answer: (6 + 2)/36, or: 2/9]

   (c) sum is a number greater than 7 but less than 11? [Answer: (5 + 4 + 3)/36, or: 1/3]

   (d) sum is a number ≥ 2 but < 12? [Answer: (36 - 1)/36, or: 35/36]

(2) What answers would you give to the questions of Exercise 1 above if you threw two dice of the <u>same</u> color? [Answer: the same answers]

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If a class is particularly interested in probability problems, and if you wish to go further into the subject, there are several ways in which you can proceed. [The 24th Yearbook of the National Council of Teachers of Mathematics contains David A. Page's chapter on probability which is full of excellent suggestions for classroom activities. See also the two references given on TC[4-G]c.]

One way is to ask students to compute other sums in Exercise 8. A student can give the probability of any sum if he is patient enough to find all of the points in his $6 \times 6 \times 6$ cube which gives him that sum. Therefore, the real quest should be for short cuts in locating points with a certain sum.

Another direction in which you can go more deeply into this work on probability is to help students discover some of the elementary rules for combining probabilities. If you throw a red die and a white die

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together, the number which you obtain from one of the dice is not affected by the number which you obtain from the other. Thus, for example, it is just as likely that you will get 2 on the white die whether you get 6 on the red die or not. We say that "getting 2 on the white die" and "getting 6 on the red die" are <u>independent events</u>. The student has learned enough about the probabilities with dice to assert that the probability of each of these two independent events is 1/6 and the probability of the occurrence of both of these events together is 1/36. Here the student has an illustration of the rule that the probability of <u>both</u> of two independent events is the product of the probabilities of each of the events separately. For example, if you throw a coin and a die together, the probability of getting heads on the coin is 1/2 and the probability of getting 1 on the die is 1/6. These two events are independent. [We decide this intuitively because we <u>just know</u> that the outcome for the coin does not affect and is not affected by the outcome for the die.] Following the rule, we find that the probability of getting heads on the coin <u>and</u> 1 on the die is $(1/2) \times (1/6)$, or 1/12. You can see that this kind of analysis gives more rapid results than the method the student has been using where he would say that the equally likely events are

(h, 1),	(h, 2),	(h, 3),	(h, 4),	(h, 5),	(h, 6),
(t, 1),	(t, 2),	(t, 3),	(t, 4),	(t, 5),	(t, 6),

where 'h' stands for heads, 't' stands for tails, and the numerals stand for the outcome for the die. Since there are 12 of these equally likely events, the student assigns probability 1/12 to each of them. He finds that only one of these equally likely events gives him the outcome "heads on the coin, 1 on the die", and he comes to the same conclusion as he did when using the rule.

Here is another fundamental idea. If you consider throwing one die, there are six outcomes which are equally likely, and therefore are each assigned probability 1/6. Moreover, you know, without even feeling the necessity to specify it, that no two of these events can occur simultaneously. So, the probability of both of these events occurring is 0. Such a pair of events, where the occurrence of one of the events precludes the occurrence of the other event, is called a pair <u>of mutually exclusive events</u>.

To get a clearer picture of what is meant by 'mutually exclusive events', let us consider an example of two events which are <u>not</u> mutually exclusive. Suppose event A is the event of getting 5 in <u>one</u> throw of a die. [The probability of A is 1/6.] Suppose event B is the event of getting

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a number different from 4 in one throw of a die. [The probability of B is 5/6.] Now, what is the probability of both A and B occurring in one throw of a die? Certainly it is possible for both A and B to occur in one throw. In fact, we can say that both A and B have occurred if and only if we get 5 in one throw. So, the probability of both A and B is 1/6. Take a second example. What is the probability of getting in one throw both an even number and a number which is different from 4? Both events will have occurred when we get either 2 or 6. So, the probability of both events is 2/6. [But, the probability of getting in one throw both 5 and an even number is 0. These events are mutually exclusive.]

The importance of mutually exclusive events is that their probabilities "add up". The student has been using this rule intuitively throughout these exercises. For example, he assumed that since the probability with 2 dice of (2, 4) is 1/36 and since the probability of (1, 3) is 1/36, the probability of either (2, 4) or (1, 3) is (1/36) + (1/36), or 1/18. We get correct results by adding the probabilities because the probability of getting in a single throw of 2 dice both (2, 4) and (1, 3) is 0. You know the probability of both events is 0 because such an outcome is simply impossible.

Here is another example. In throwing one coin and one die, the probability of

(A) getting the pair (h, 1) is 1/12,

(B) getting the pair (h, 4) is 1/12, and

(C) getting a tail without regard for the die is 1/2.

Every pair, A and B, A and C, B and C, of these events is a pair of mutually exclusive events. Therefore, we can conclude that the probability of getting

tails on the coin

or

heads on the coin and 1 on the die

or

heads on the coin and 4 on the die

is (1/2) + (1/12) + (1/12), or 2/3.

You can check this result by going back to the 12 equally likely events and seeing that 8 of them correspond with the compound event of "getting A or B or C".

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Here is a short quiz [for class work or take-home] which can be used to maintain skills of earlier units.

Simplify.

1. $\frac{-15a}{-3}$ 2. $m + 5 + n - 2m + 7 + 6n$ 3. $(-3.2)(-5a)$

4. $(56a^2b^2) \div (-4a)$ 5. $\frac{2}{3}x - \frac{1}{5}y + \frac{3}{2}x - 2y$ 6. $\frac{c}{21} \times \frac{3d}{4c}$

7. $4(a + 3b) + 5(3a - b) - 7(-2a + 5b)$

8. $6x(2y - 3x) - 5y(y - 2x) + x - 3y^2$

Expand.

9. $(x - 3)(x + 7)$ 10. $(a - 4)(a + 4)$

11. $(3b - 5)(2b + 1)$ 12. $(2 - 3x)(2x + 5)$

Factor.

13. $4ax - 12bx^2$ 14. $2a^2x^2 - 3a^2y^2 + 9a^2z$

15. $x^2 + 5x - 14$ 16. $6a^2 - 12ab - 18b^2$

Solve.

17. $3(x - 3) + 7 = 4 - 5(2x - 4)$ 18. $\frac{3}{x} + 2 = \frac{9}{x} - \frac{9}{2x}$

19. $40 + x = x(x - 2)$

20. Towns A and B are 375 miles apart. One car starts from A for B at the same time as another starts from B for A. They pass each other at a point C on the road between the towns. If the car which started at A averaged 34 miles per hour between A and C, and the car from B averaged 41 miles per hour between B and C, how far is C from A?

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Answers for quiz.

1. $5a$ 2. $12 + 7n - m$ 3. $16a$

4. $-14ab^2$, $[a \neq 0]$ 5. $\frac{13}{6}x - \frac{11}{5}y$ 6. $\frac{d}{28}$, $[c \neq 0]$

7. $33a - 28b$ 8. $-18x^2 + 22xy + x - 8y^2$

9. $x^2 + 4x - 21$ 10. $a^2 - 16$ 11. $6b^2 - 7b - 5$

12. $10 - 11x - 6x^2$ 13. $4x(a - 3bx)$ 14. $a^2(2x^2 - 3y^2 + 9z)$

15. $(x + 7)(x - 2)$ 16. $6(a - 3b)(a + b)$ 17. 2

18. $3/4$ 19. $8, -5$ 20. 170 miles

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A lattice [as we use the word] is a set which is the cartesian product of "discretely ordered" sets. Such a cartesian product can easily be pictured by means of a rectangular array of dots. The number of columns in the array is the same as the number of elements in the first "factor" of the cartesian product, and the number of rows is the same as the number of elements in the second factor of the cartesian product.

The lattice to be pictured in this case is the cartesian product of $\{-2, -1, 0, 1, 2, 3\}$ by $\{-2, -1, 0, 1, 2\}$. Assigning the elements in the first factor to the columns in a left-to-right order in accordance with the < - ordering of the elements is purely conventional. Similarly, the assignment of elements in the second factor to rows in the "upward" direction, in accordance with the < - ordering among elements is also conventional. The assignment of ordered pairs of the cartesian product to dots in the array as described on page 4-2 is conventional. These conventions which determine how the rectangular array of dots is to be used in picturing a lattice enhance the usefulness of the picture in studying properties of the lattice.

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Notice at the bottom of page 4-2 the use of the word 'graph'. As in the case of a real number with respect to a picture of the number line, the graph of an ordered pair of real integers is a dot on a picture of the lattice. Each component of the ordered pair is a coordinate of the dot.

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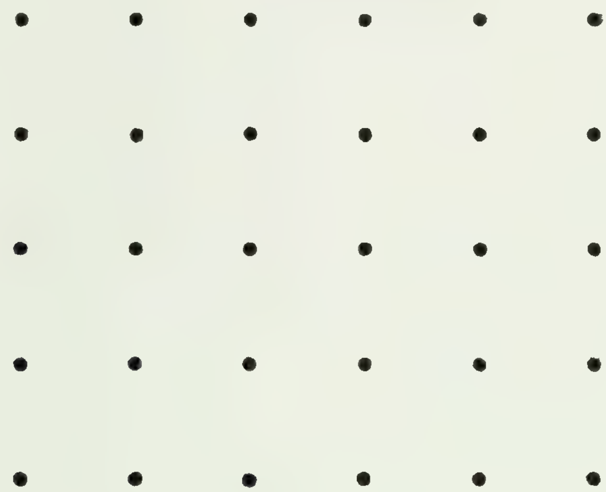
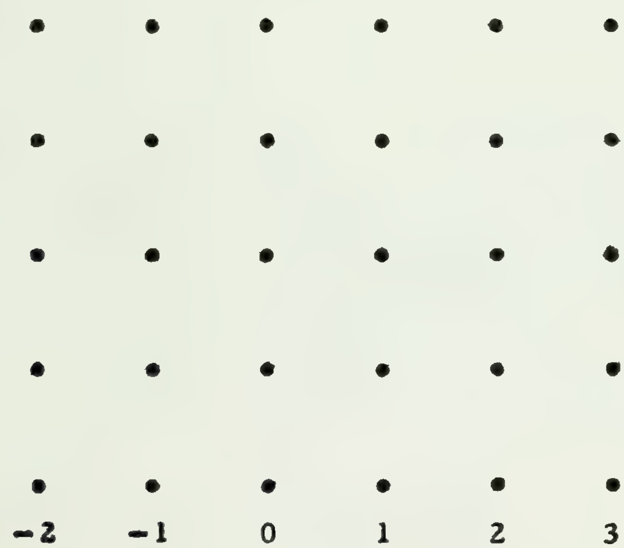
Short quizzes are given on the pages listed below.

TC[4-1, 2]b	TC[4-10]b	TC[4-15]d
TC[4-19]g	TC[4-23, 24]c	TC[4-27, 28]b
TC[4-32]f, g	TC[4-41, 42]c	TC[4-86]

4.01 Lattices.--The rectangular arrays of dots with which you worked in the preceding section are pictures of lattices. Each dot in such a picture corresponds with an ordered pair of numbers. Here is a picture of a lattice.



An easy way to assign ordered pairs of numbers to these dots is to number the columns and the rows. Pick one of the columns and label it with a '0'. Then label the columns to the right of this one with a '1', a '2', a '3', and so on, and label the columns to the left of the 0-column with a '−1', a '−2', etc.



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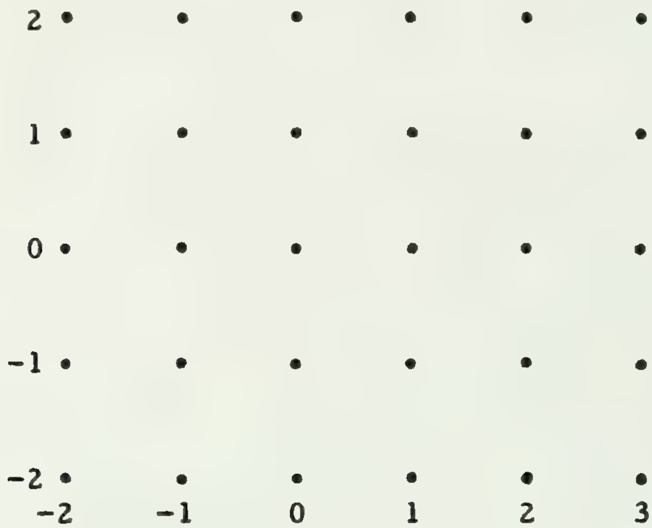
Next, pick one of the rows as the 0-row, and label the rows above it with a '1', a '2', and so on, and label the rows below it with a '-1', a '-2', and so on.



Now, we can assign an ordered pair of numbers to each of these 30 dots by agreeing that the first component of the ordered pair tells the column the dot is in and that the second component tells the row the dot is in. So, for example, the ordered pair (3, 2) corresponds with the dot in the upper right hand corner. Point to the dot which corresponds with (1, 2). With (2, 1). With (0, 0). With (0, 2). With (2, 3).

Each of the 30 dots in the picture is the <u>graph</u> of the corresponding ordered pair of numbers, and the first and second components of each of the 30 ordered pairs are the <u>first coordinate</u> and <u>second coordinate</u> of the corresponding dot.

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You may want to precede the exercises in Part A by asking students how many dots are in the rectangular array, how many ordered pairs are pictured, and to describe the cartesian product which is pictured by this rectangular array of dots.

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Answers for Part A.

1.	A: (2, 4)	B: (3, 0)	C: (−1, 3)	D: (0, 2)
	E: (5, 5)	F: (3, −2)	G: (−2, 2)	H: (5, 3)
	I: (−2, −1)	J: (0, 0)	K: (−3, −1)	

2.	A: 2,	B: 3,	C: −1,	D: 0,	E: 5,	F: 3,
	G: −2,	H: 5,	I: −2,	J: 0,	K: −3	
3.	A: 4,	B: 0,	C: 3,	D: 2,	E: 5,	F: −2,
	G: 2,	H: 3,	I: −1,	J: 0,	K: −1	

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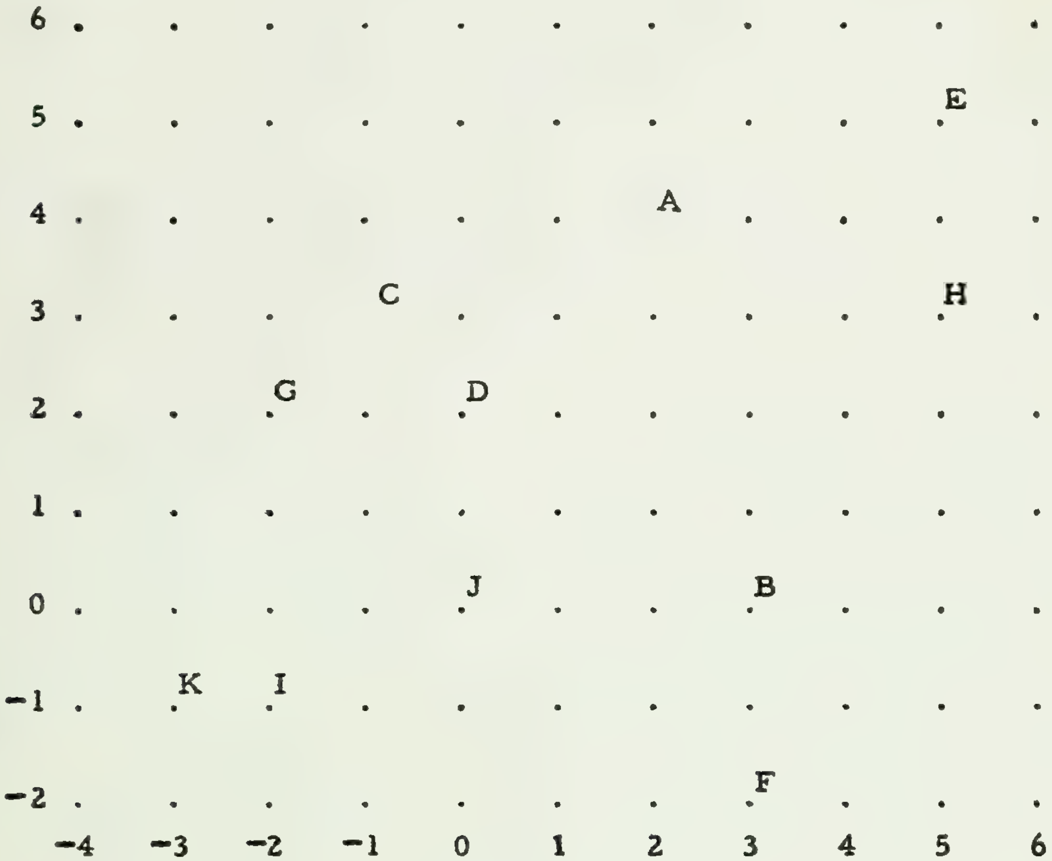
Exercises 5 through 9 should be done in class so that the teacher can move about among the students and check the work. This is the best time to correct any misinterpretations of the exercises.

EXERCISES

A. Study this picture of a lattice, and answer the questions which follow.



1. Give the ordered pairs of numbers which correspond with the eleven labeled dots. [Sample. A: (2, 4)]

2. Give the first coordinate of each labeled dot.

3. Give the second coordinate of each labeled dot.

4. How many labeled dots in the picture have first coordinate equal to second coordinate?

5. Label with an 'L' the graph of the ordered pair (4, 3).

6. Label with an 'M' the dot whose first coordinate is 3 and whose second coordinate is 4.

7. Draw a dashed line which connects all the dots which have first coordinate 0.

8. Draw a dashed line which connects all the dots which have second coordinate 0.

9. Label with a 'W' the graph of (0, 1).

B. Consider the cartesian product

$$\{-3, -2, -1, 0, 1, 2, 3\} \times \{-5, -4, -3, -2, -1, 0, 1, 2, 3\}.$$

Make a picture of this lattice.

Tell how many dots are graphs of ordered pairs with

(1) first component 2 .

(2) second number -3 .

(3) first number greater than or equal to 2 .

(4) second component less than or equal to -1 .

(5) first number greater than 1 <u>and</u> second number less than 2 .

(6) first number greater than 1 <u>or</u> second number less than 2 .

(7) first component equal to second component .

(8) first number 1 more than second number .

(9) second number twice first number .

(10) second component equal to 3 more than 2 times first component .

(11) second number 5 more than 2 times first number <u>and</u> with second number 3 more than first number .

[Supplementary exercises are in Part A on page 4-118.]

THE NUMBER PLANE LATTICE

Imagine the set of all the ordered pairs whose first and second components are <u>integral real numbers</u> [for short: <u>real integers</u>], that is, the numbers

$$\ldots\ -3,\ -2,\ -1,\ 0,\ +1,\ +2,\ +3,\ \ldots\ .$$

This set is the cartesian product of the set of real integers by itself [that is, the <u>cartesian square</u> of the set of real integers]. We call this cartesian product <u>the number plane lattice</u>. ['plane' because we picture it as a flat surface like a sheet of paper, and flat surfaces are called <u>planes</u>.]

The exercises in Part B prepare the students for more conventional exercises in graphing. The student who tells you, in answer to Exercise (1), that there are 9 dots with first component 2, has gone a long way toward understanding why the graph of 'x = 2' is a line, not just a single point. [Naturally, you should not bring in any "x, y-language" at this time.]

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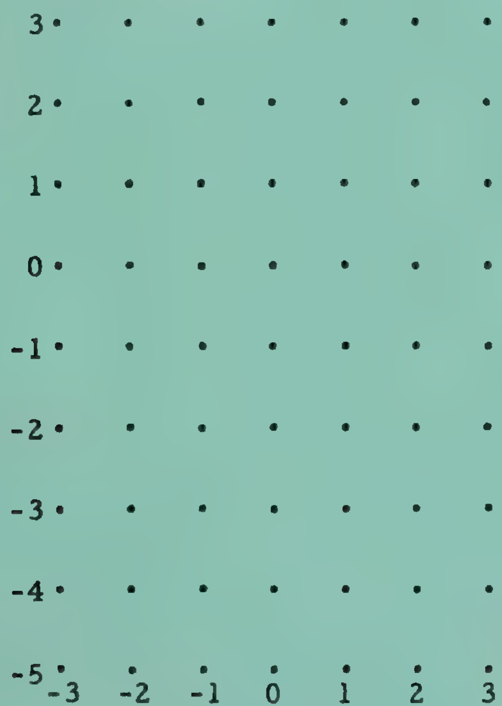
Answers for Part B.

Here is a picture of the lattice.



(1) 9 (2) 7 (3) 18 (4) 35 (5) 14

[The dots whose coordinates satisfy the requirement in question (5) are the graphs of (2, 1), (2, 0), (2, −1), (2, −2), (2, −3), (2, −4), (2, −5), (3, 1), (3, 0), (3, −1), (3, −2), (3, −3), (3, −4), (3, -5). [This exercise is in preparation for a later exercise in drawing the graph of the sentence 'x > 1 and y < 2'.]]

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(6) 53 [The dots whose coordinates satisfy the requirement in question (6) are all the dots in the (2)-column, the (3)-column, the (1)-row, the (0)-row, the (−1)-row, the (−2)-row, the (−3)-row, the (−4)-row, and the (−5)-row. There are 18 dots in the two columns; the dots in the rows total 49. But the 7 rows include 14 of the 18 dots in the 2 columns. So, since we don't count any dot twice, the total number of dots which satisfy the requirement of the problem is 18 + 49 - 14, or 53.]

(7) 7 [The graphs of (−3, −3), (−2, −2), (−1, −1), (0, 0), (1, 1), (2, 2), (3, 3).]

(8) 7 [The graphs of (−3, −4), (−2, −3), (−1, −2), (0, −1), (1, 0) (2, 1), (3, 2).]

(9) 4 [The graphs of (−2, −4), (−1, −2), (0, 0), (1, 2).]

(10) 4 [The graphs of (−3, −3), (−2, −1), (−1, 1), (0, 3).]

[Exercise 11 may be difficult for some students. Do not give your students any formal methods. In reading the two conditions, they may feel that something is wrong and that we must have intended this to be two problems. Point out that because of the 'and' they should hunt for points where both conditions are satisfied. If the student is baffled, remind him that if he picks a point, he can decide whether the components of that point are numbers which satisfy the conditions given in Exercise 11. And, he can try every point in the lattice if necessary! Students will probably come up with the idea that the first condition leads to one "line" of points and that the second condition leads to another "line" of points. To meet both conditions, a point must be in both sets.]

(11) 1 [The graph of (−2, 1).]

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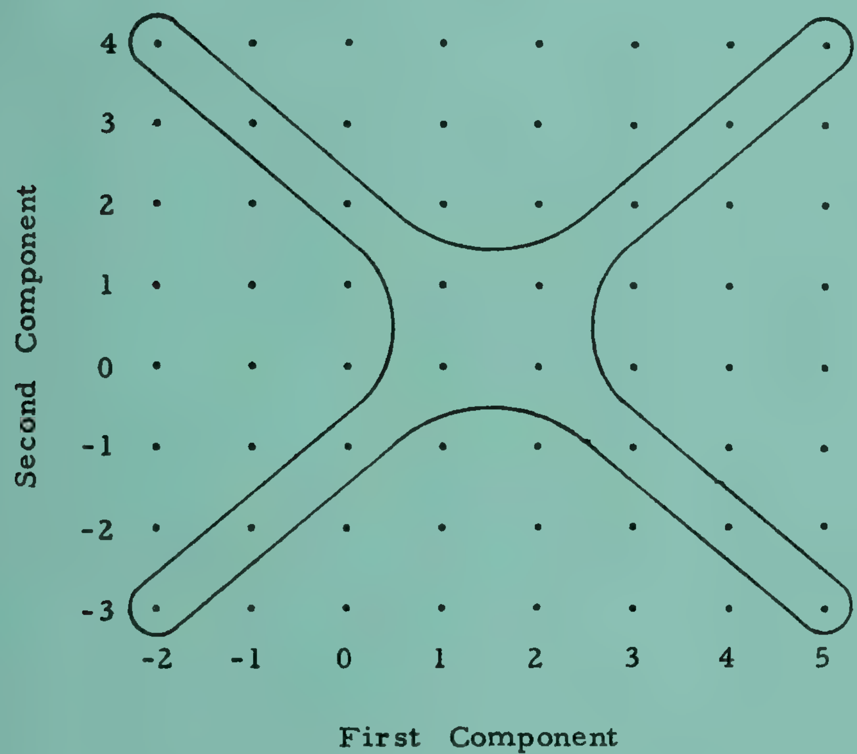
After discussing the exercises on page 4-4, you may want to experiment with this. Before class, draw this figure on the board:



Ask the class, "What expression, or expressions, could you write which would serve as a "rule" for finding all the points contained in the cross marked on the lattice?"

Some will probably suggest (using 'f' for 'first component' and 's' for 'second component'):

$$f = 1 + s \quad \underline{\text{or}} \quad f + s = 2.$$

Then you will need to consider with the class whether it is correct to use 'or'. Ask whether 'and' is a better word to use.

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Then consider these possibilities:

(1) Find all the points such that f = 1 + s,
<u>or</u>
find all the points such that f + s = 2.

(2) Find all the points such that f = 1 + s <u>or</u> f + s = 2.

(3) Find all the points such that f = 1 + s,
<u>and</u>
find all the points such that f + s = 2.

(4) Find all the points such that f = 1 + s <u>and</u> f + s = 2.

Get the class to recognize the different implications of these sentences.

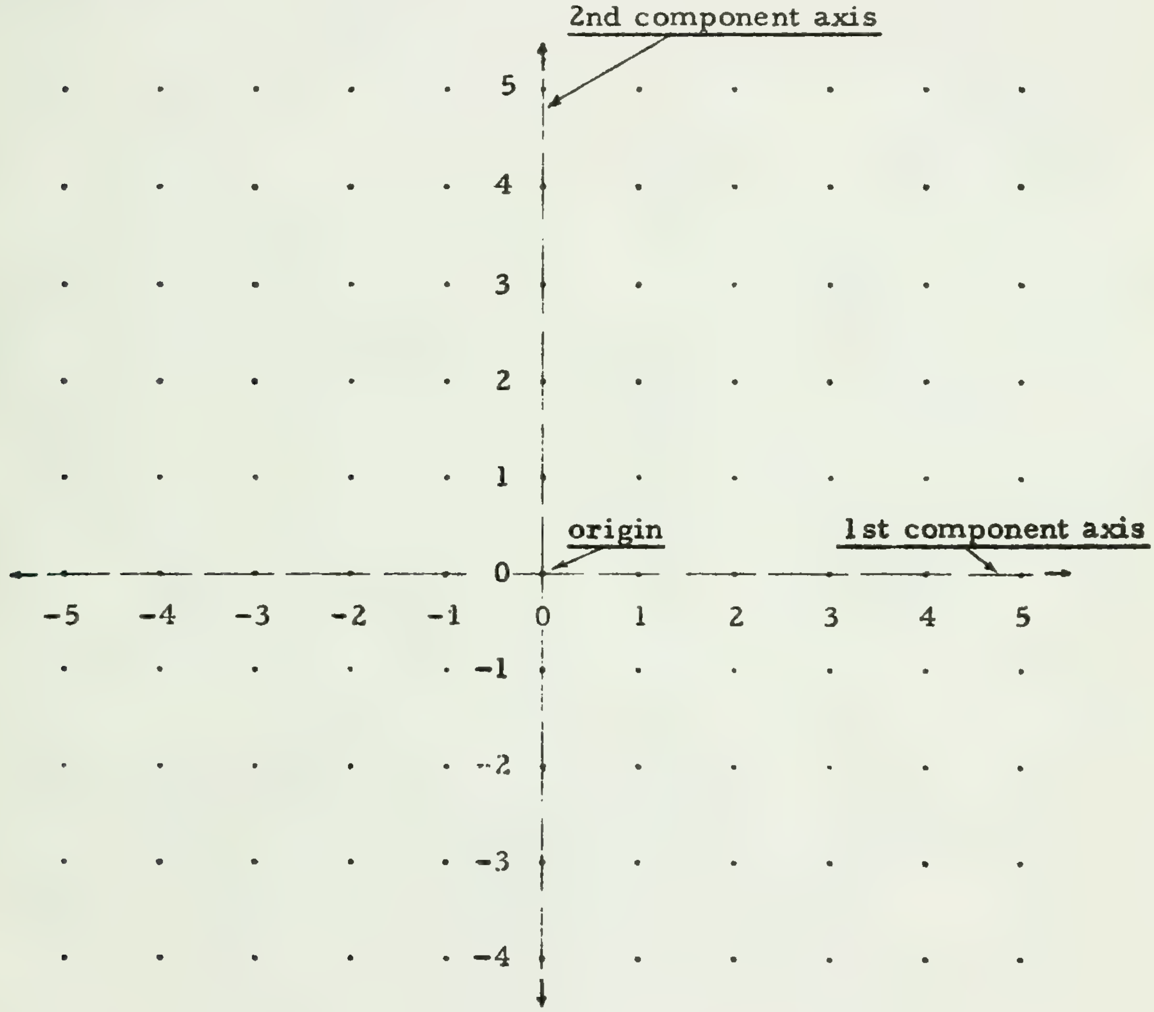
*

The cartesian square of the set of real integers contains infinitely many ordered pairs. This cartesian product is called 'the number plane lattice' and, as in the case of the number line, we can picture only a finite part of the lattice. The small arrows on the picture [on page 4-5] indicate the infinite nature of the lattice. [Later, when there is no possibility for confusion, we shall omit the arrows.] [It is customary to picture a part of the lattice which contains (0, 0). Sometimes this leads to misconceptions. We have included the exercises in Part B on page 4-23 to correct these.]

[4-4]

B

Just as you could draw pictures of only part of the number line, it is possible to draw pictures of only part of the number plane lattice. Still, we shall call such pictures pictures of the number plane lattice. When drawing such a picture, you may first select a 0-column and a 0-row. [It's helpful if you draw a dashed line connecting the dots in the 0-column and a dashed line connecting the dots in the 0-row.] Then, number the rows and columns by writing numerals close to the dots in the 0-column and 0-row. You can put little arrows at the ends of this column and this row to show that you have a picture of only part of the lattice.



The dots in a picture of the number line correspond with points of the number line, that is with real numbers; the dots in a picture of the number plane lattice correspond with points of the number plane lattice, that is, with ordered pairs of real integers. The process of locating the dot which corresponds with a given ordered pair is called graphing the ordered pair or plotting the point.

The set of points in the lattice each of which has second component 0 is called the <u>first component axis</u>. The dots in the picture which correspond with these points are lined up horizontally; they are the dots which have 0 as second coordinate. What is meant by <u>the second component axis</u>? The first and second component axes have one point in common, and this point is called <u>the origin</u>. What ordered pair is this point?

EXERCISES

<u>A</u>. Draw a picture of the number plane lattice. Plot the points listed below and label them with the given letters.

A: (3, 5)	B: (2, 5)	C: (−3, 1)	D: (2, 0)
E: (0, 0)	F: (0, −2)	G: (6, 1)	H: (1, 6)
I: (−3, −4)	J: (4, −3)	K: (10, −10)	L: (−9, −9)

<u>B</u>. Draw a picture of the number plane lattice [your diagram should contain enough dots so that you can plot the points (−8, 8), (−8, −8), (8, 8), and (8, −8).] Draw the sets of points described below. Mark the dots in some particular fashion so that you can tell the sets apart. [We abbreviate 'real integers' to 'integers'.]

1. The set of all ordered pairs of integers with first component equal to 1 less than the second component.

2. The set of all ordered pairs of integers with first component 3 more than second component.

3. The set of all ordered pairs of integers such that the sum of the components of each ordered pair is 9.

4. The set of all ordered pairs of integers such that 8 is the sum of the second component and twice the first component.

5. The set of all ordered pairs of integers with second component 7.

6. The set of all ordered pairs of integers which correspond with dots which have first coordinate −3.

7. The set of all ordered pairs of integers such that the first component is less than −5 <u>and</u> the second component is greater than 6.

d

[To do the exercises on pages 4-6 and 4-7, and on other pages of Section 4.01, the students will need several sheets of ''lattice paper'' in order to keep the graphing manageable. We have found that usually three or four exercises per sheet is a good allocation. If you have facilities for doing mimeograph or ditto work, you will want to prepare a quantity of $8\frac{1}{2} \times 11$ sheets with dots on the whole page spaced approximately as they are in the picture on page 4-5. [If it is very difficult for you to get such work done, you can have the students use cross section paper and place dots on the intersections of the printed lines.] When the students plot two or more sets of points on the same diagram, they can indicate the points in the same set by encircling each point, putting a tiny square or triangle around each point, or by using colored pencils.]

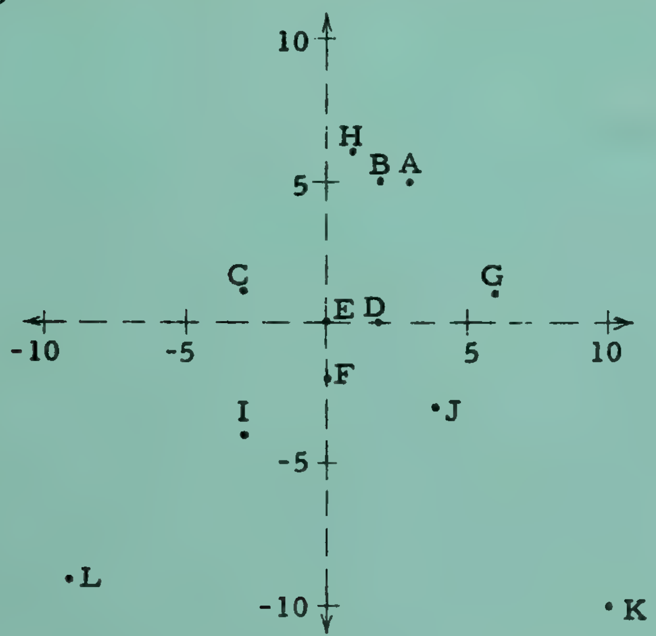
*

Note the use of the word 'point' in the first paragraph on page 4-6. The elements of a lattice are ordered pairs, and these elements are commonly called 'points'. [See TC[1-99]a and TC[1-99]b for comments on how the word 'point' is used in mathematics.] The component axes are sets of ordered pairs of numbers, not sets of dots.

*

Answers for Part A.

[This is a rough sketch to show the location of the dots.]



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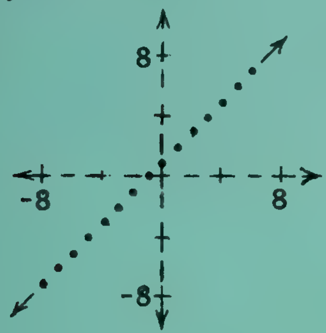
Answers for Part B.

[Again, these are rough sketches to show the location of the dots.]

1.



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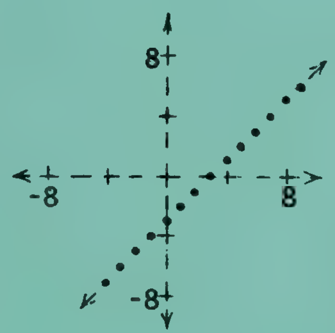

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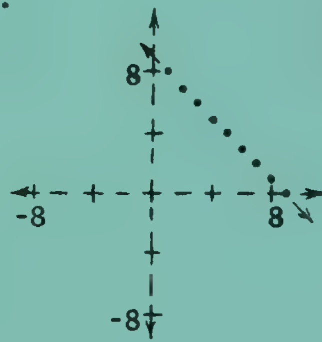

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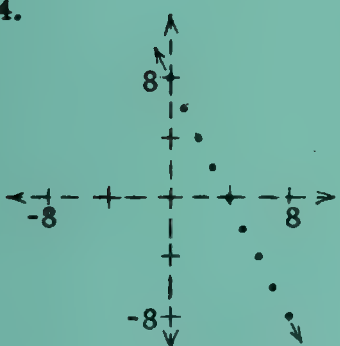

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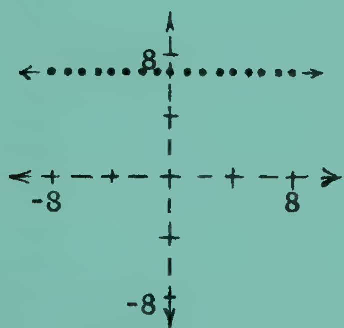

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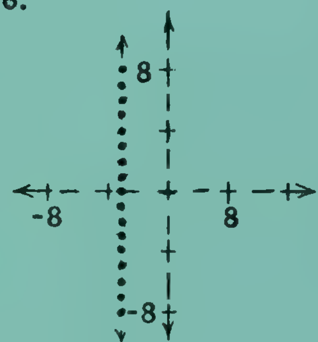

*

Exercise 7 [and 8, 9, and 10 on page 4-7] may require special attention. Consider Exercise 7. In discussing this problem at the board, the first thing you might do is to locate dots with first coordinate less than −5. Draw a light vertical line between the (-6)-column and the (−5)-column, and shade the region to the left of this vertical line. The shaded region contains the graphs of ordered pairs of integers with first component less than −5. Next, locate dots with second coordinate greater than 6 by drawing a horizontal line halfway between the (6)-row and the (7)-row,

(continued on TC[4-7])

T

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[4-7]

d

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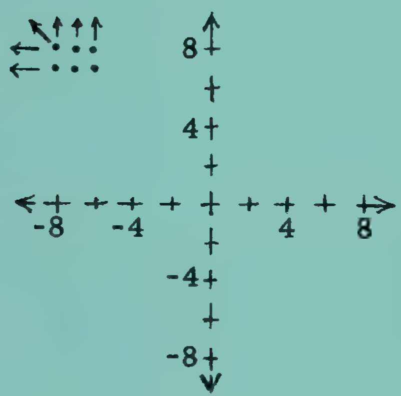
4/25

and shade the region above this line. The part of the picture common to the shaded regions contains graphs of those members of the number plane lattice with first component less than −5 and second component greater than 6. [This type of exercise prepares students for the notion of the intersection of two sets.]

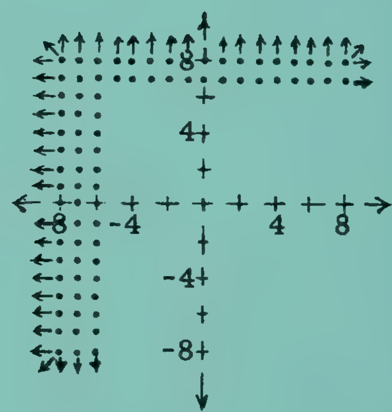
In Exercise 8, the set in question contains all the ordered pairs with first component less than −5 together with all the ordered pairs with second component greater than 6. These are the ordered pairs with first component less than −5 or second component greater than 6. [This type of exercise is preparation for the notion of the union of two sets.]

*

7.



8.



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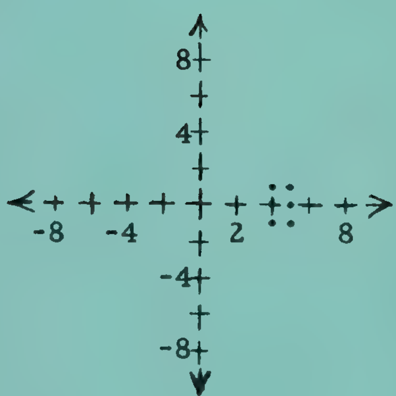
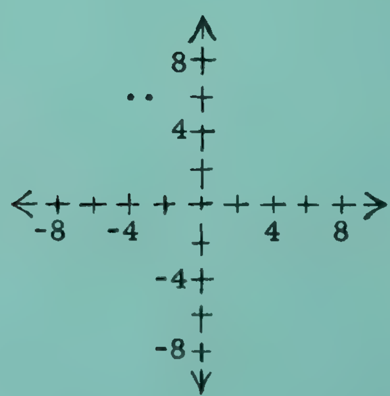


10.



8. Repeat Exercise 7 using 'or' instead of 'and'.

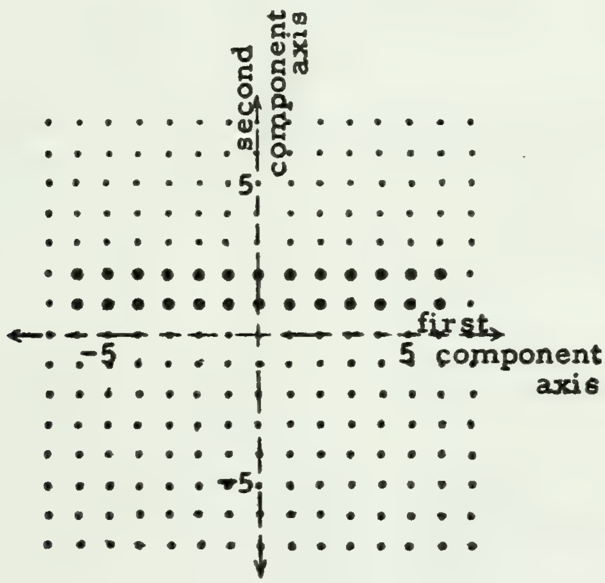
9. The set of all ordered pairs of integers such that the second component is less than 2 but greater than −2 and the first component is greater than 3 but less than 6. [How many points are there in this set?]

10. The set of all ordered pairs of integers such that the first component is greater than −5 but less than −2 and the second component is 6.

[Supplementary exercises are in Part B, pages 4-118 and 4-119.]

C. Here are pictures of sets of ordered pairs of integers. Write descriptions of the sets pictured, using the type of wording of the exercises of Part B.

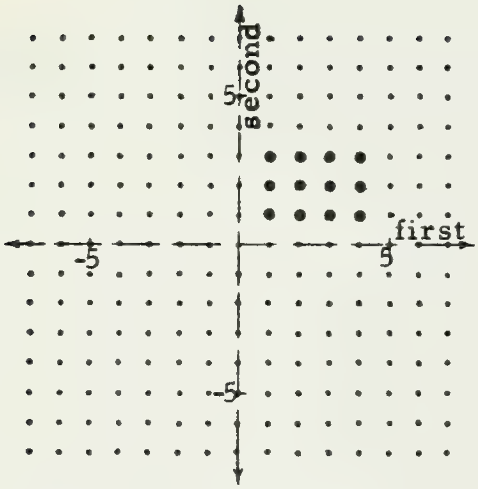
Sample.



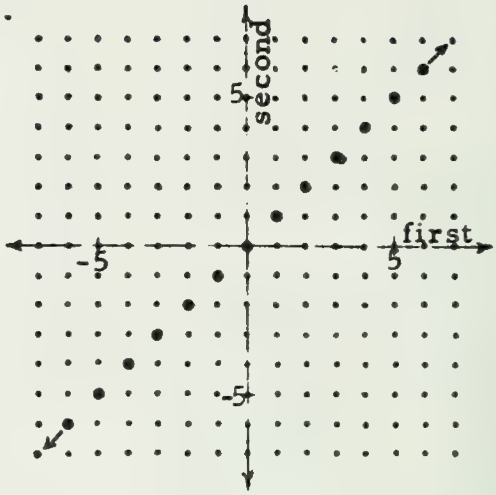
Solution. The first coordinate of each of these dots is an integer between −7 and 7; the second coordinate is either 1 or 2. So, a description of this set is: the set of all ordered pairs of integers such that the first component is greater than −7 but less than 7 and the second component is greater than 0 but less than 3.

(continued on next page)

1.



2.



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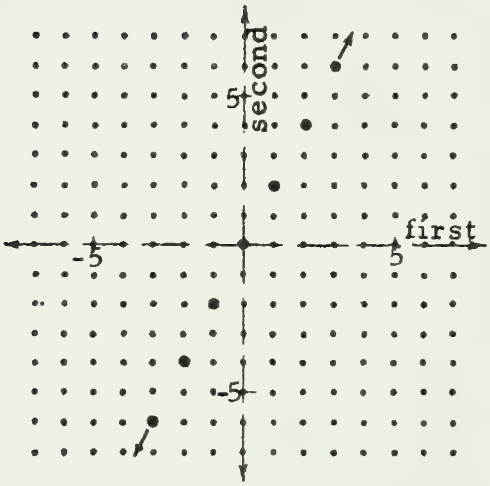

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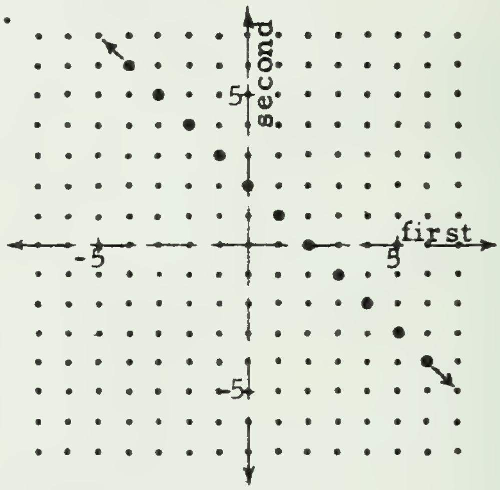

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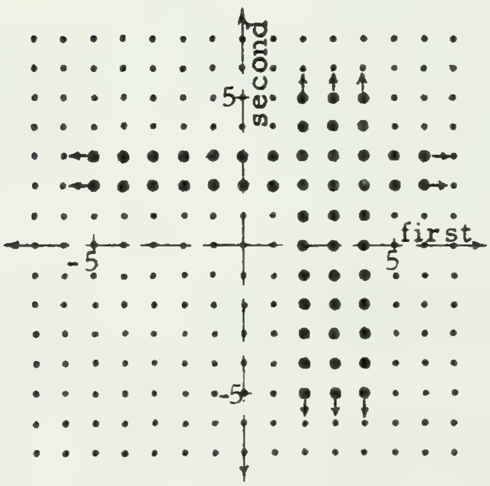

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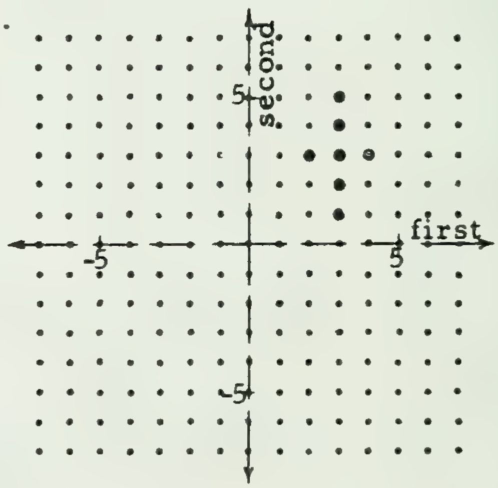


[Supplementary exercises are in Part C, pages 4-119 and 4-120.]

Answers for Part C [which begins on page 4-7].

[We give one description for each set pictured. Your students may suggest others.]

1. The set of all ordered pairs of integers such that the first component is greater than 0 but less than 5, and the second component is greater than 0 but less than 4.

2. The set of all ordered pairs of integers such that the first component is equal to the second component.

3. The set of all ordered pairs of integers such that the second component is twice the first component.

4. The set of all ordered pairs of integers such that the sum of the first component and the second component is 2.

5. The set of all ordered pairs of integers such that the first component is greater than 1 and less than 5, <u>or</u> the second component is greater than 1 and less than 4.

6. The set of all ordered pairs of integers such that (the first component is 3 and the second component is greater than 0 but less than 6) <u>or</u> (the second component is 3 and the first component is greater than 1 but less than 5). [You may find it interesting to write this description on the board, then ask your class to name the ordered pairs of integers which belong to the set described without looking at the picture in the text. After these pairs are listed on the board, <u>then</u> have students check them with the picture. For your convenience, here is a list of the ordered pairs.

   (2, 3), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (4, 3).]

[4-8]

1.

[4-9]

Answers for Part D.

1. (a) $\{(x, y),$ x and y integers: $x = 3 + y\}$ [Ex. 2]

   (b) $\{(x, y),$ x and y integers: $x + y = 9\}$ [Ex. 3]

   (c) $\{(x, y),$ x and y integers: $2x + y = 8\}$ [Ex. 4]

   (d) $\{(x, y),$ x and y integers: $y = 7\}$ [Ex. 5]

   (e) $\{(x, y),$ x and y integers: $x = -3\}$ [Ex. 6]

   (f) $\{(x, y),$ x and y integers: $x < -5$ and $y > 6\}$ [Ex. 7]

   (g) $\{(x, y),$ x and y integers: $x < -5$ or $y > 6\}$ [Ex. 8]

   (h) $\{(x, y),$ x and y integers: $3 < x < 6$ and $-2 < y < 2\}$ [Ex. 9]

   (i) $\{(x, y),$ x and y integers: $-5 < x < -2$ and $y = 6\}$ [Ex. 10]

   [There is more than one way to write set selectors in these descriptions. For example, in (a) we could have written '$y = x - 3$' instead of '$x = 3 + y$'. We have given them here as we think the students will probably give them.]

2. (a) $\{(x, y),$ x and y integers: $0 < x < 5$ and $0 < y < 4\}$ [Ex. 1]

   (b) $\{(x, y),$ x and y integers: $x = y\}$ [Ex. 2]

   (c) $\{(x, y),$ x and y integers: $y = 2x\}$ [Ex. 3]

   (d) $\{(x, y),$ x and y integers: $x + y = 2\}$ [Ex. 4]

   (e) $\{(x, y),$ x and y integers: $1 < x < 5$ or $1 < y < 4\}$ [Ex. 5]

   (f) $\{(x, y),$ x and y integers: $(x = 3$ and $0 < y < 6)$ or $(y = 3$ and $1 < x < 5)\}$ [Ex. 6]

   [Again, there is more than one way to write set selectors in these descriptions. For (d), someone may suggest '$y = -x + 2$' instead of '$x + y = 2$'.]

* * *

As you have probably noticed, these word-descriptions of sets can be quite complicated. We can simplify the job of describing sets by using the notation introduced in Unit 3. For example, consider the description given in the first exercise of Part B:

> The set of all ordered pairs of integers
> with first component equal to 1 less
> than the second component.

We use a pair of braces, '{' and '}', to show that we are talking about a set, an index of pronumerals such as '(x, y)' or '(p, q)' to show that we are talking about ordered pairs of real numbers, a restriction to indicate what kind of real numbers we are interested in, and an open sentence which is satisfied by those ordered pairs (and only those) which belong to the set in question. Thus, the set described in words above is also described by:

$$\{(x, y),\ x \text{ and } y \text{ integers}:\ x = y - 1\}.$$

[Read this as 'the set of ordered pairs of integers such that the first component is 1 less than the second component' or, more simply, as 'the set of ordered pairs (x, y) of integers such that x = y - 1'.]

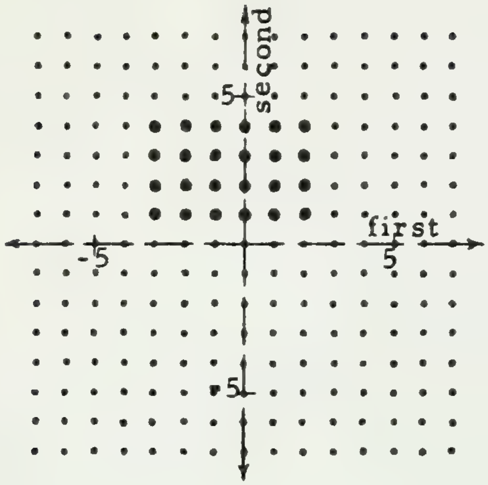
* * *

D. 1. Rewrite the word-descriptions of Exercises 2-10 of Part B using the brace-notation mentioned above.

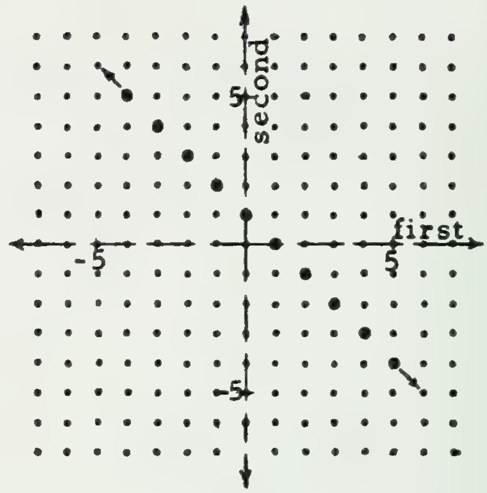
2. Rewrite the descriptions of the sets pictured in the exercises of Part C.

3. Describe the sets pictured on page 4-10.

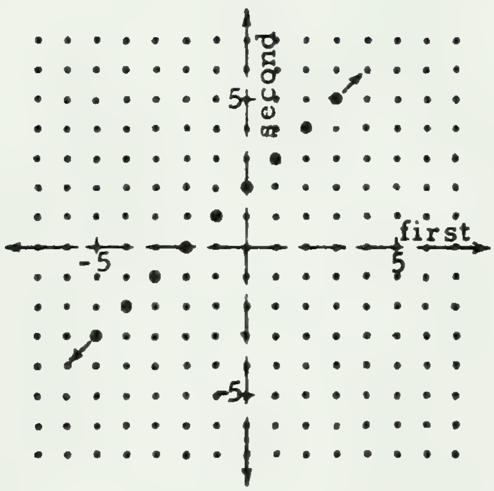
(a)



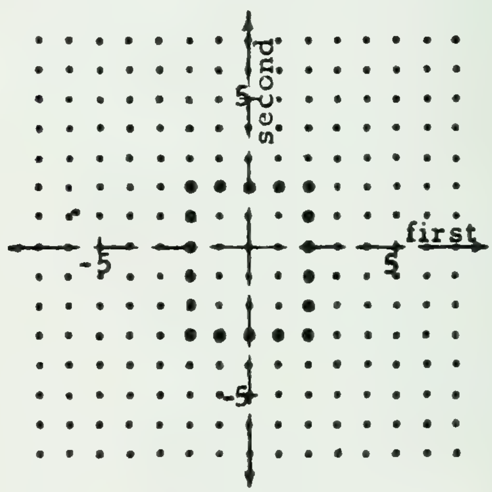
(b)



(c)



(d)



[Supplementary exercises are in Part D on page 4-120.]

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3\. (a) $\{(x, y),\ x \text{ and } y \text{ integers}:\ -4 < x < 3 \text{ and } 0 < y < 5\}$ [Ex. 1]

(b) $\{(x, y),\ x \text{ and } y \text{ integers}:\ x + y = 1\}$ [Ex. 2]

(c) $\{(x, y),\ x \text{ and } y \text{ integers}:\ y = x + 2\}$ [Ex. 3]

(d) $\{(x, y),\ x \text{ and } y \text{ integers}:\ [-4 < y < 3 \text{ and } (x = -2 \text{ or } x = 2)] \text{ or } [-3 < x < 3 \text{ and } (y = -3 \text{ or } y = 2)]\}$ [Ex. 4]

[Exercise 3(d) should be quite interesting to your better students. Challenge them to write other descriptions, using brace-notation, for the set pictured. Here are two other descriptions.

$$\{(x, y),\ x \text{ and } y \text{ integers}:\ [-4 < y < 3 \text{ and } |x| = 2] \text{ or } [-3 < x < 3 \text{ and } |y + \tfrac{1}{2}| = 2\tfrac{1}{2}]\}$$

$$\{(x, y),\ x \text{ and } y \text{ integers}:\ [|x| = 2 \text{ or } y = -3 \text{ or } y = 2] \text{ and } [-3 < x < 3 \text{ and } -4 < y < 3]\}$$

A student who gives either of these has done some good thinking!]

*

[4-10]

(a

Review Quiz.

1. Which of the following sentences is an instance of the commutative principle for addition?

   (a) $(3 + 7) + (12 + 9) = (3 + 7 + 12) + 9$

   (b) $(12 + 3 + 7) + 9 = 12 + (3 + 7 + 9)$

   (c) $(7 + 9) + (12 + 3) = (12 + 3) + (7 + 9)$

   (d) $7 + (9 + 3) + 12 = 7 + 9 + (3 + 12)$

2. For each number of arithmetic $h \neq 0$, and for each number of arithmetic $m \neq 0$, if one pound of apples costs $3/h$ cents then $5/m$ pounds of apples cost ___________ cents.

3. There are two numbers whose average is 21. One of the numbers is -17. What is the other number?

4. Which of the following sentences is true?

   (a) $\sqrt{6^2 + 10^2} = 6 + 10$ (b) $\sqrt{(6 + 10)^2} = 6 + 10$

   (c) $\sqrt{6^2 - 10^2} = (6 - 10)(6 + 10)$ (d) $\sqrt{6^2 \times 10^2} = (6 \times 10)^2$

   (e) $\sqrt{6 - 10} = (6 - 10)^2$

5. Below are listed pairs of numbers. Insert '>' or '<' between the two numerals in a pair so that the resulting statement is true.

   (a) $-\frac{5}{8} \quad -\frac{5}{9}$ (b) $\frac{5}{6} \quad \frac{7}{8}$ (c) $\frac{4}{5} \quad -\frac{14}{15}$

   (d) $\frac{8}{9} \quad \frac{10}{11}$ (e) $-\frac{187}{452} \quad -\frac{186}{453}$

*

Answers for Quiz.

1. (c) 2. $15/(hm)$ 3. 59 4. (b)

5. (a) < (b) < (c) > (d) < (e) <

(a

[4-11]

(a'

Some of your students may want to say that the union of A and B con-contains 18 points because "you should count some of the points twice". If they do this, they are confusing the

union of two sets

with the

sum of two numbers.

These are very different ideas. A particular point cannot "belong to a set twice". Either it belongs or it does not belong.

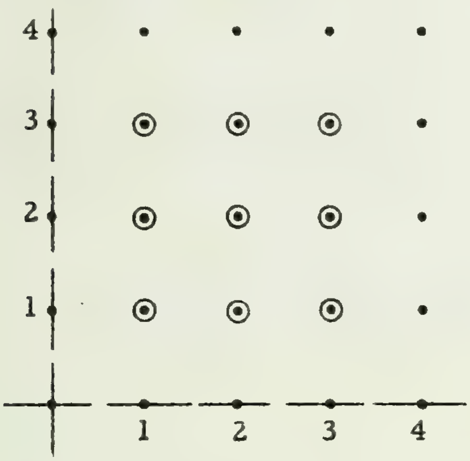
A playful teaching method we have used runs like this:

> Go to each point and ask it, "Do you belong to either of these two sets?" If it says 'yes', take it, if it says 'no', don't take it. When you have asked every point, you will have the union of the two sets in question. <u>If you ask a point the question several times, you are doing extra work but it won't change the result.</u>

It so happens that if the intersection of two sets is empty, then the union of the two sets has a number of elements which is the sum of the number of elements in each set.

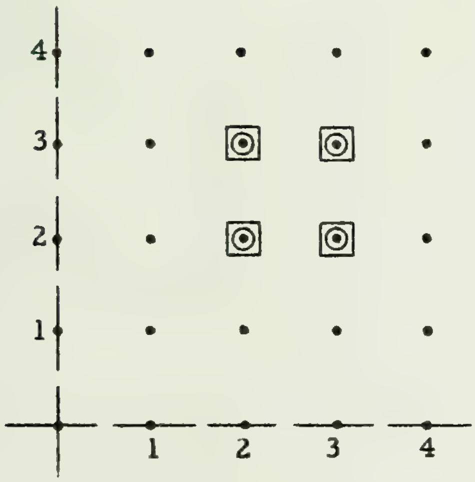
INTERSECTIONS AND UNIONS

Below are four pictures of the same part of the number plane lattice. Each picture shows a set. The first picture shows set A, the picture in the upper right shows set B, the picture in the lower left shows the set consisting of all the points which belong to both A and B, and the fourth picture shows the set of all points which belong either to A or to B.

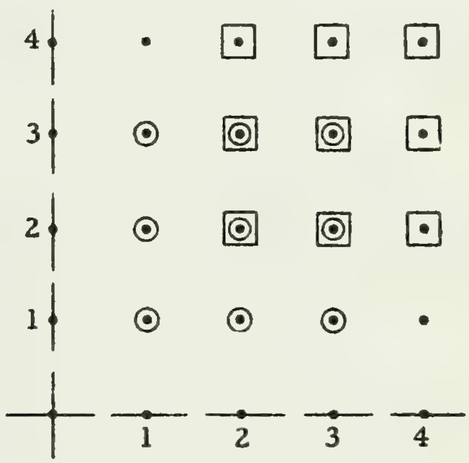


Set A
(9 points)

Set B
(9 points)



Intersection of
Set A and Set B
(4 points)



Union of
Set A and Set B
(14 points)

Notice the words <u>intersection</u> and <u>union</u>. The intersection of two sets is the set which consists of the elements which belong to both of the given sets. The union of two sets is the set which consists of the elements which belong to either of the given sets.

In the illustration on page 4-11, A and B each have 9 points. The intersection of A and B consists of the 4 points each of which belongs to both A and B. The union of A and B consists of the 14 points each of which belongs to at least one of the sets A and B.

EXERCISES

<u>A</u>. In each of the following exercises you are given descriptions of a set A and a set B. For each exercise,

(a) plot the points in each set on the same diagram,

(b) tell the number of points in each set,

(c) tell the number of points in the intersection, and

(d) tell the number of points in the union.

1. $A = \{(x, y),\ x \text{ and } y \text{ integers}:\ -2 < x < 3 \text{ and } 3 < y < 6\}$

   $B = \{(x, y),\ x \text{ and } y \text{ integers}:\ 0 < x < 6 \text{ and } -1 < y < 5\}$

   the number of points in A = ________

   the number of points in B = ________

   the number of points in the intersection of A and B = ________

   the number of points in the union of A and B = ________

2. $A = \{(x, y),\ x \text{ and } y \text{ integers}:\ -7 < x < -2 \text{ and } -4 < y < 3\}$

   $B = \{(x, y),\ x \text{ and } y \text{ integers}:\ -8 < x < 0 \text{ and } -4 < y < 0\}$

   n(A) = ____________ ['n(A)', read as 'en of A', means the number of elements in set A.]

   n(B) = ________

   n(the intersection of A and B) = ________

   n(the union of A and B) = ________

In other books dealing with the elementary algebra of sets, you may find 'sum' or 'join' used as a synonym for 'union', and 'product' or 'meet' as a synonym for 'intersection'.

*

Note the abbreviation 'n(A)' introduced in Exercise 2. 'n(A)' may be read as 'the number of elements in A' or, more briefly, as 'en of A'.

*

We introduce two symbols in Exercise 7 on page 4-13. '$A \cap B$' is read as 'the intersection of sets A and B', or 'A intersection B'. '$\cap$' is called 'the intersection sign' or 'cap'. Similarly, '$A \cup B$' is read as 'the union of sets A and B', or as 'A union B'. '$\cup$' is called 'the union sign' or 'cup'.

In Unit 5 the students will make a more formal study of the operations of unioning and intersecting. At that time, they will learn, among other things, that these operations are commutative and associative, and that each is distributive with respect to the other. At the present time, all we want of the student is that he continue practicing with the ideas of unioning and intersecting, and that he become acquainted with the operation signs. [A helpful mnemonic device is that '$\cup$' reminds one of the first letter in 'union'.]

*

You can expect many students to discover that the number of elements in the union of two sets is the number of elements in one of them plus the number of elements in the other minus the number of elements in their intersection. A careful formulation of this generalization is:

For each set A, for each set B,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Note that 'A' and 'B' are not pronumerals. They are pronouns of a different sort--marks which hold places for names of <u>sets</u>.

*

The use of '^{3}' and '^{4}' as exponent symbols occurs for the first time in Exercise 12. Students should be able to guess their meanings. The use of such exponent symbols is studied at greater length on pages 4-59ff.

*

[4-12]

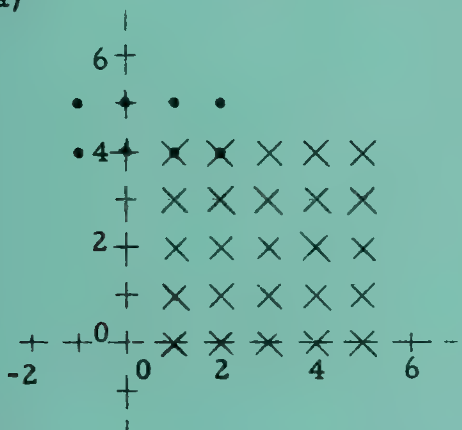
We think your students will really enjoy doing this group of exercises. Our pilot school teachers have reported much enthusiasm here. The first eight exercises usually make a comfortable assignment. Exercises 9, 10, and 13 are a bit more difficult. Spiral the last ten exercises into subsequent assignments.

*

Answers for Part A [on pages 4-12, 4-13, 4-14, and 4-15].

[In the sketches which follow, we have used '•'s to indicate the points of set A and '×'s to indicate the points of set B. The points in the intersection of sets A and B will therefore be indicated by '⨯'s.]

1. (a)

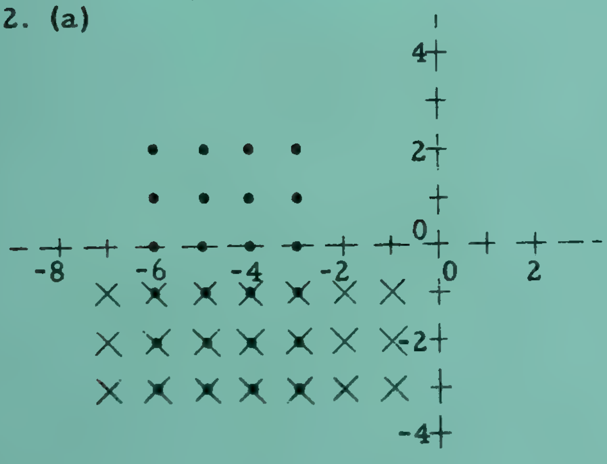


(b) the number of points in A = 8
the number of points in B = 25

(c) the number of points in the intersection of A and B = 2

(d) the number of points in the union of A and B = 31

2. (a)



(b) $n(A) = 24$, $n(B) = 21$

(c) $n(A \cap B) = 12$

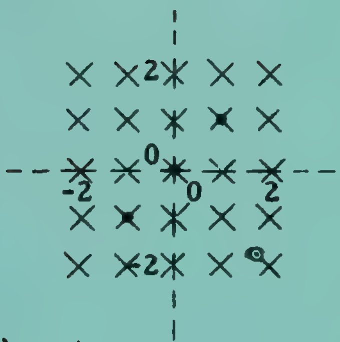
(d) $n(A \cup B) = 33$

s

[4-13]

s

7. (a)

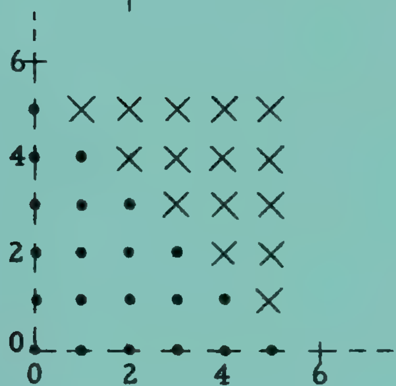


(b) $n(A) = 3$, $n(B) = 25$

(c) $n(A \cap B) = 3$

(d) $n(A \cup B) = 25$

8. (a)

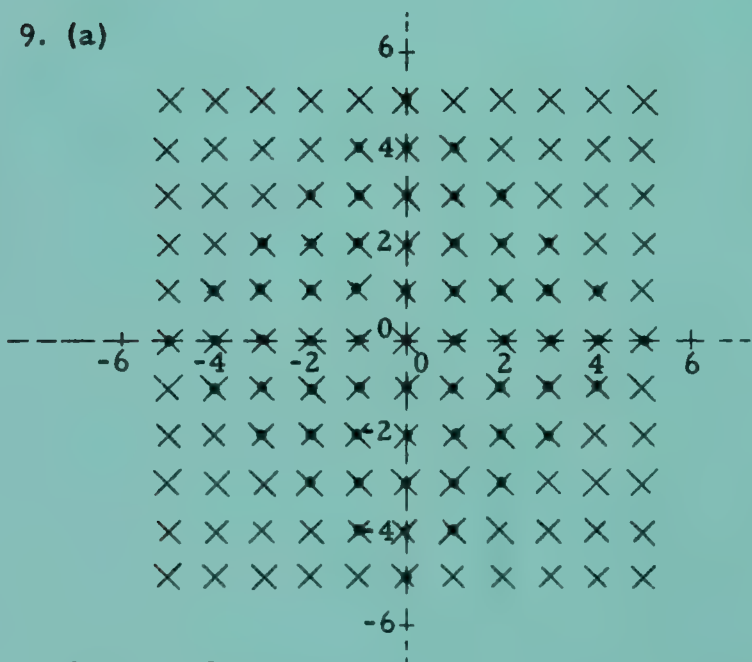


(b) $n(A) = 21$, $n(B) = 15$

(c) $n(A \cap B) = 0$

(d) $n(A \cup B) = 36$

9. (a)



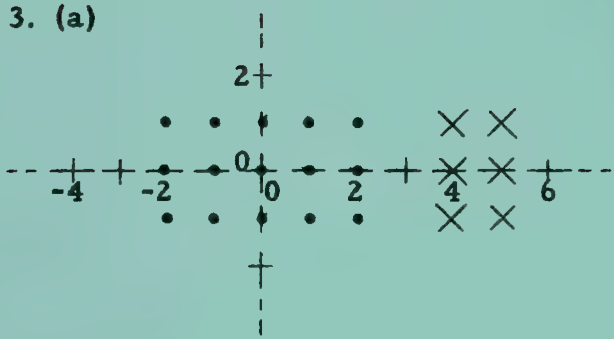
(b) $n(A) = 61$, $n(B) = 121$

(c) $n(A \cap B) = 61$

(d) $n(A \cup B) = 121$

[4-13]

3. (a)

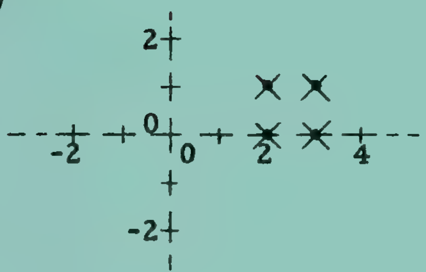


(b) $n(A) = 15$, $n(B) = 6$

(c) $n(A \cap B) = 0$

(d) $n(A \cup B) = 21$

4. (a)

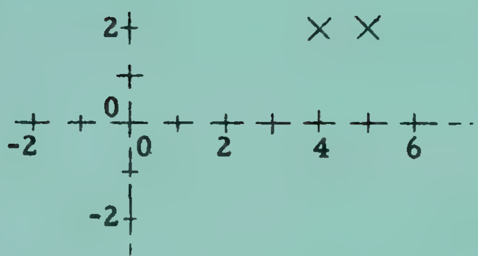


(b) $n(A) = 4$, $n(B) = 4$

(c) $n(A \cap B) = 4$

(d) $n(A \cup B) = 4$

5. (a)



(b) $n(A) = 0$ [set A is $\emptyset$], $n(B) = 2$

(c) $n(A \cap B) = 0$

(d) $n(A \cup B) = 2$

6. (a) Set $A = \emptyset$ and set $B = \emptyset$; so there are no points to plot!

(b) $n(A) = 0$, $n(B) = 0$; (c) $n(A \cap B) = 0$; (d) $n(A \cup B) = 0$

3. $A = \{(x, y),$ x and y integers: $|x| < 3$ and $|y| < 2\}$

$B = \{(x, y),$ x and y integers: $3 < x < 6$ and $|y| < 2\}$

n(A) = ________, n(B) = __________

n(the intersection of A and B) = _______, n(the union of A and B) = _____

4. $A = \{(2, 0), (2, 1), (3, 0), (3, 1)\}$

$B = \{(x, y),$ x and y integers: $1 < x < 4$ and $-1 < y < 2\}$

n(A) = __________, n(B) = __________

n(the intersection of A and B) = ______, n(the union of A and B) = _____

5. $A = \{(x, y),$ x and y integers: $-4 < x < -3$ and $-2 < y < 0\}$

$B = \{(x, y),$ x and y integers: $3 < x < 6$ and $1 < y < 3\}$

n(A) = __________, n(B) = __________

n(the intersection of A and B) = ______, n(the union of A and B) = _____

6. $A = \{(x, y),$ x and y integers: $-1 < x < 0$ and $5 < y < 6\}$

$B = \{(x, y),$ x and y integers: $5 < x < 6$ and $-1 < y < 0\}$

n(A) = __________, n(B) = __________

n(the intersection of A and B) = _____, n(the union of A and B) = _____

7. $A = \{(x, y),$ x and y integers: $x = y$, $|x| < 2$, and $|y| < 2\}$

$B = \{(x, y),$ x and y integers: $|x| < 3$ and $|y| < 3\}$

n(A) = __________, n(B) = __________

$n(A \cap B)$ = __________, $n(A \cup B)$ = __________

['$A \cap B$', read as 'A intersection B', means the intersection of sets A and B;

'$A \cup B$', read as 'A union B', means the union of sets A and B.]

(continued on next page)

8. $A = \{(x, y), \text{ x and y integers}: x + y \leq 5, x \geq 0, \text{ and } y \geq 0\}$

$B = \{(x, y), \text{ x and y integers}: x + y > 5, x < 6, \text{ and } y < 6\}$

$n(A) =$ ____, $n(B) =$ ____, $n(A \cap B) =$ ____, $n(A \cup B) =$ ____

9. $A = \{(x, y), \text{ x and y integers}: |x| + |y| \leq 5\}$

$B = \{(x, y), \text{ x and y integers}: |x| \leq 5\} \cap$

$\{(x, y), \text{ x and y integers}: |y| \leq 5\}$

$n(A) =$ ____, $n(B) =$ ____, $n(A \cap B) =$ ____, $n(A \cup B) =$ ____

10. $A = \{(x, y), \text{ x and y integers}: -3 < x < 0 \text{ or } 3 < y < 6\}$

$B = \{(x, y), \text{ x and y integers}: 2 < x < 6 \text{ or } -4 > y > -7\}$

[Note that neither A nor B is <u>finite</u>, that is, you can't count the elements in these sets. Sets which are not finite are called <u>infinite sets.</u> When you are asked the number of elements in a set which is not finite, just answer that the question doesn't make sense because the set is infinite. [In a later course you may learn about other numbers which can be used to tell how many elements there are in an infinite set.]]

$n(A) =$ ____, $n(B) =$ ____, $n(A \cap B) =$ ____, $n(A \cup B) =$ ____

11. $A = \{(x, y), \text{ x and y integers}: x + y = 0\}$

$B = \{(x, y), \text{ x and y integers}: x - y = 0\}$

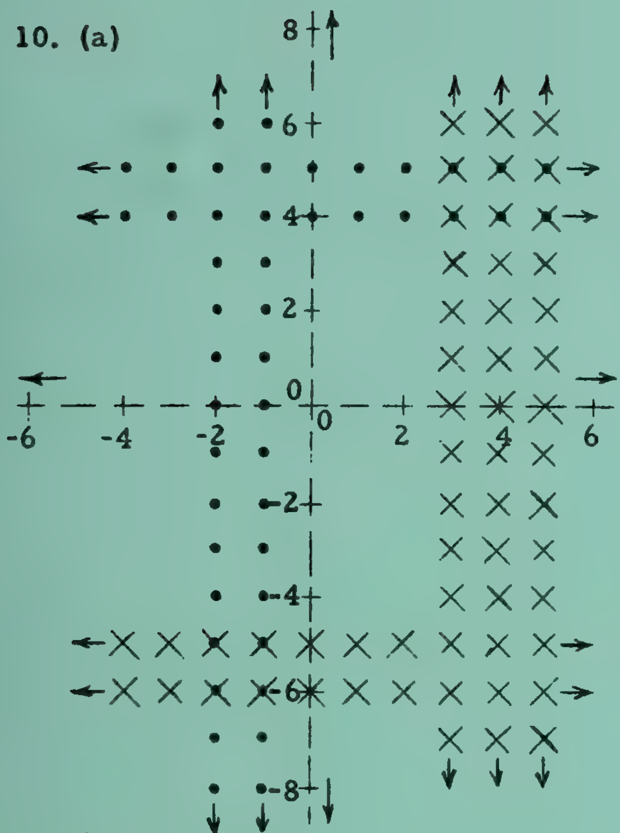
$n(A) =$ ____, $n(B) =$ ____, $n(A \cap B) =$ ____, $n(A \cup B) =$ ____

12. $A = \{(x, y), \text{ x and y integers}: 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\}$

$B = \{(x, y), \text{ x and y integers}: y = x, y = x^2, y = x^3, \text{ and } y = x^4\}$

$n(A) =$ ____, $n(B) =$ ____, $n(A \cap B) =$ ____, $n(A \cup B) =$ ____

10. (a)



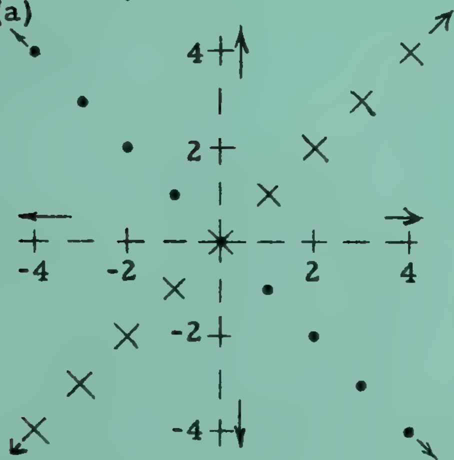
(b) n(A) = ___
n(B) = ___
These two questions do not make sense because each set is infinite.

(c) $n(A \cap B) = 10$

(d) $n(A \cup B) =$ ___ This question does not make sense because the set is infinite.

[Note the use of the word 'or' in these descriptions in Exercise 10 and compare its use with that of the word 'and' in Exercise 8, for example. The word 'or' gives you a union; 'and' gives you an intersection.]

11. (a)

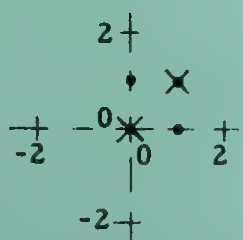


(b) n(A) = ___
n(B) = ___
These two questions do not make sense because each set is infinite.

(c) $n(A \cap B) = 1$

(d) $n(A \cup B) =$ ___ This question does not make sense because the set is infinite.

12. (a)



(b) $n(A) = 4$, $n(B) = 2$

(c) $n(A \cap B) = 2$

(d) $n(A \cup B) = 4$

[4-14]

8

[4-15]

Review Quiz.

Simplify.

1. $3x(2 - z) - 2z(3 - x)$
2. $(5x - 3y) - \frac{1}{2}(4x + 8y)$
3. $(x - 2)^2 + (x - 3)^2$
4. $(2y + 7)^2 - (3 - 2y)^2$
5. $\frac{1}{2x} + \frac{3}{x} - \frac{5}{3x}$
6. $\frac{4 - x}{3 - y} + \frac{9}{3 - x}$

Solve.

7. $8x - 2(3 - 4x) + 7 = 0$
8. $y = 11 - \frac{30}{y}$
9. Mrs. Ames is considering investing a total of $80,000 in two businesses. One of the enterprises will pay her 6% on her investment but is somewhat more risky than the other which pays only 3%. What is the least amount of money she can invest in the 6% enterprise [with the rest invested in the 3% one] so that she can receive a total annual income of $3450 from the two?

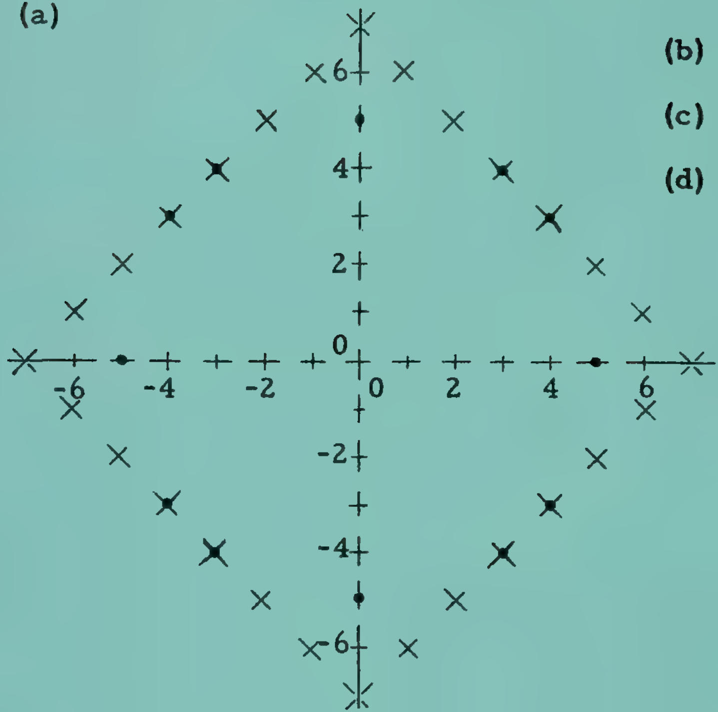
☆10. Two boy scouts hiked from 3 p.m. to 9 p.m. one evening, covering the same route going and returning. Their hiking rate is 4 miles per hour on the level, 3 miles per hour uphill, and 6 miles per hour downhill. If the total distance covered on the level is 8 miles, what is the total distance covered on hills?

*

Answers for Quiz.

1. $6x - xz - 6z$
2. $3x - 7y$
3. $2x^2 - 10x + 13$
4. $40y + 40$
5. $\frac{11}{6x}$, $[x \neq 0]$
6. $\frac{39 - 7x + x^2 - 9y}{(3 - y)(3 - x)}$, $[y \neq 3 \neq x]$
7. $-\frac{1}{16}$
8. 6, 5
9. $35000
10. 16 miles [Total time for the hike was 6 hours; 2 hours were spent hiking 8 miles on the level. So, 4 hours remained for walking up and down hills. If d is the distance uphill [and downhill!] then d/3 is the time spent walking uphill, and d/6 is the time spent walking downhill. $(d/3) + (d/6) = 4$] [An interesting adaptation of this problem is to omit the datum about the number of miles walked on level ground, and to ask for the total number of miles walked.]

17. (a)



(b) $n(A) = 12$, $n(B) = 28$

(c) $n(A \cap B) = 8$

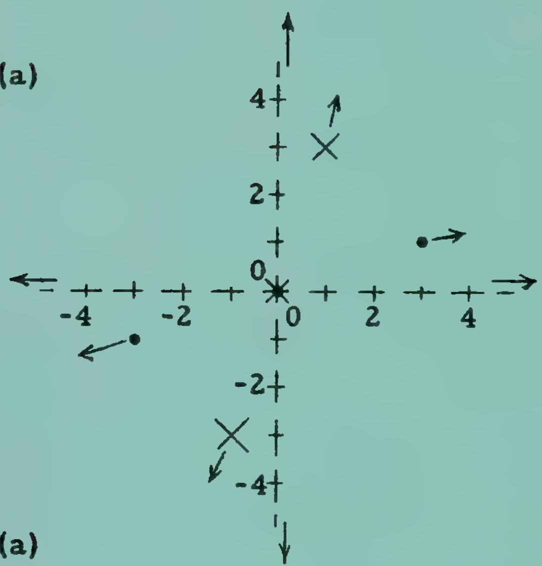
(d) $n(A \cup B) = 32$

18. (a) Set $A = \emptyset$ and set $B = \emptyset$ so there are no points to plot!

(b) $n(A) = 0$, $n(B) = 0$; (c) $n(A \cap B) = 0$; (d) $n(A \cup B) = 0$

[4-15]

15. (a)

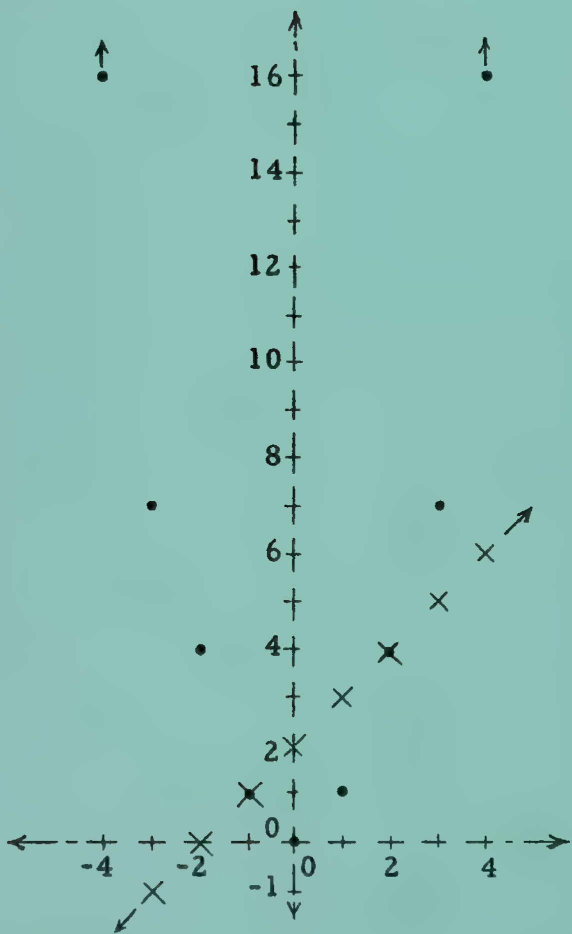


(b) A and B are infinite sets.

(c) $n(A \cap B) = 1$

(d) $A \cup B$ is an infinite set.

16. (a)



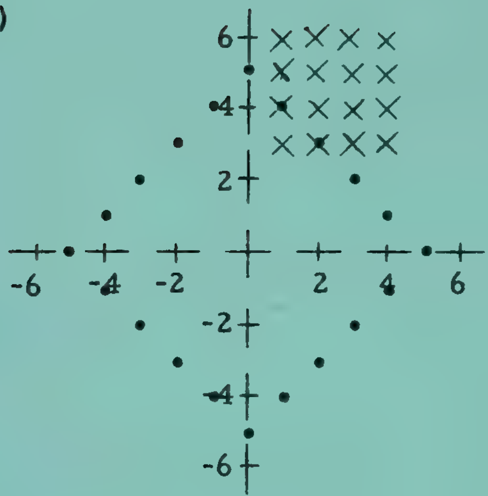
(b) A and B are infinite sets.

(c) $n(A \cap B) = 2$

(d) $A \cup B$ is an infinite set.

[4-15]

13. (a)

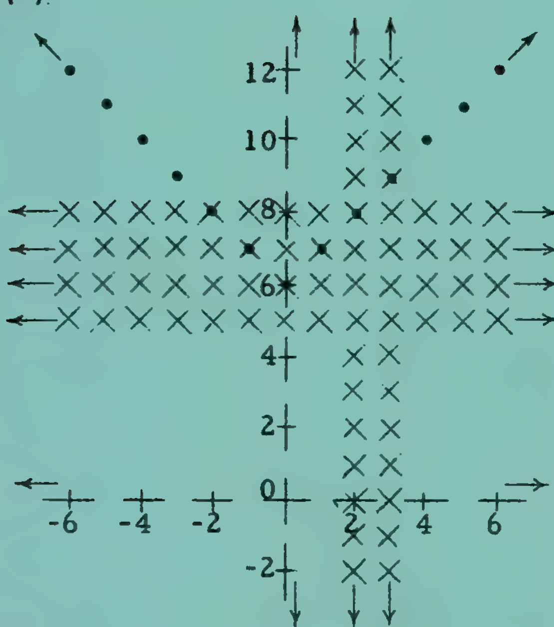


(b) n(A) = 20, n(B) = 16

(c) $n(A \cap B) = 2$

(d) $n(A \cup B) = 34$

14. (a)



(b) A and B are infinite sets

(c) $n(A \cap B) = 6$

(d) $A \cup B$ is an infinite set

[You may need to remind the class that '$|x|$' is an abbreviation for '$^{+}|x|$'. Have students consider some ordered pairs which belong to set A. After a few are listed [e. g., (0, 6), (3, 9), (−3, 9)], ask for the ordered pair in A which has the smallest second component. After establishing that the second component must be ≥ 6, it is easy to find other ordered pairs in A. Just fill the blanks in:

(___, 7), (___, 8), (−1, ___), (−2, ___), ...]

[Set B is really the union of $\{(x, y),\ x \text{ and } y \text{ integers}:\ 2 \leq x < 4\}$ and $\{(x, y),\ x \text{ and } y \text{ integers}:\ 4 < y < 9\}$.]

13. $A = \{(x, y),\ x \text{ and } y \text{ integers}:\ |x| + |y| = 5\}$

$B = \{(x, y),\ x \text{ and } y \text{ integers}:\ 5 > x > 0 \text{ and } 7 > y > 2\}$

$n(A) =$ ____, $n(B) =$ ____, $n(A \cap B) =$ ____, $n(A \cup B) =$ ____

14. $A = \{(x, y),\ x \text{ and } y \text{ integers}:\ y - |x| = 6\}$

$B = \{(x, y),\ x \text{ and } y \text{ integers}:\ 2 \leq x < 4 \text{ or } 4 < y < 9\}$

$n(A) =$ ____, $n(B) =$ ____, $n(A \cap B) =$ ____, $n(A \cup B) =$ ____

15. $A = \{(m, n),\ m \text{ and } n \text{ integers}:\ m = 3n\}$

$B = \{(m, n),\ m \text{ and } n \text{ integers}:\ n = 3m\}$

$n(A) =$ ____, $n(B) =$ ____, $n(A \cap B) =$ ____, $n(A \cup B) =$ ____

16. $A = \{(r, s),\ r \text{ and } s \text{ integers}:\ s = r^2\}$

$B = \{(p, q),\ p \text{ and } q \text{ integers}:\ q = p + 2\}$

$n(A) =$ ____, $n(B) =$ ____, $n(A \cap B) =$ ____, $n(A \cup B) =$ ____

17. $A = \{(x, y),\ x \text{ and } y \text{ integers}:\ x^2 + y^2 = 25\}$

$B = \{(x, y),\ x \text{ and } y \text{ integers}:\ |x| + |y| = 7\}$

$n(A) =$ ____, $n(B) =$ ____, $n(A \cap B) =$ ____, $n(A \cup B) =$ ____

18. $A = \{(x, y),\ x \text{ and } y \text{ integers}:\ 2y + 2x = 1\}$

$B = \{(x, y),\ x \text{ and } y \text{ integers}:\ 3y - 3x = 1\}$

$n(A) =$ ____, $n(B) =$ ____, $n(A \cap B) =$ ____, $n(A \cup B) =$ ____

[Supplementary exercises are in Part E on page 4-121.]

B. Number Plane Lattice Games

A number plane lattice game consists of a series of moves from point to point of the lattice with each move made according to a given rule.

Sample 1. Rule: A move from (x, y) takes you to $(x + 1, y + 1)$.

Start at $(-4, -3)$ and make 5 moves. Where do you finish?

Solution. First move: From $(-4, -3)$ to $(-4 + 1, -3 + 1)$, or $(-3, -2)$.

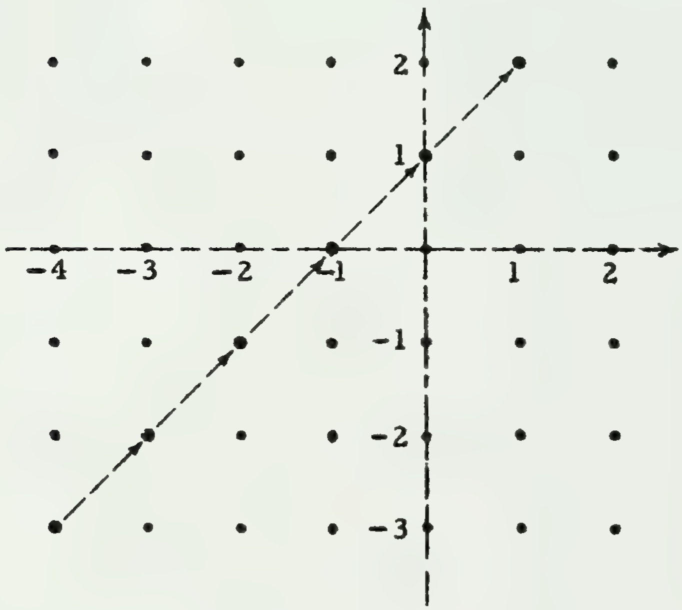
Second move: From $(-3, -2)$ to $(-3 + 1, -2 + 1)$, or $(-2, -1)$.

Third move: From $(-2, -1)$ to $(-2 + 1, -1 + 1)$, or $(-1, 0)$.

Fourth move: From $(-1, 0)$ to $(0, 1)$.

Fifth move: From $(0, 1)$ to $(1, 2)$.

So, starting at $(-4, -3)$, after five moves you reach $(1, 2)$. Here is a picture showing the moves.



Part B is another kind of exercise designed to familiarize the student with ordered pairs and graphs. Techniques become habituated as the student immerses himself in the "game"--pleasant ways of teaching and learning. [A more formal discussion of these games is given in the COMMENTARY for Exercise 11 on page 4-89.]

*

Sample 2, Sample 3, Exercise 10, and Exercise 11 prepare students for the notion of the converse of a relation. [A relation is a set of ordered pairs.] They should develop the geometric intuition that the rule:

$$(x, y) \rightarrow (y, x)$$

takes a point across the 45°-line [$\{(x, y) : y = x\}$] to its mirror image. [These matters are considered in greater detail in Unit 5.]

*

One of our pilot school teachers reported that when his class did the game described in Sample 2, someone raised the question as to whether there was a "pattern". After some discussion of this, one student pointed out that "if the picture were folded on its x = y-line, one set of dots would fall on top of the other set of dots".

*

On the day this set of exercises is to be assigned, it is a good idea to use some class time having the students work Sample 1 together, using a different starting point, and making 3, 4, and 5 moves. If you have a graph board in your room, someone can picture the moves on the board. Some of our pilot school teachers reported using peg board and vari-colored pegs with great success.

*

You may want to ask the students to make pictures showing the moves for the first few exercises. We predict that most students will soon tire of the physical act of plotting points and will arrive at final positions by strictly arithmetic (or algebraic) procedures. Do not rush them into this latter technique, however. In those exercises for which the student is asked to describe the new set of points [beginning with Exercise 9], he may find the picture of the moves helpful in writing such descriptions, particularly in view of his experience in Part D on page 4-9.

B.

Sample 2. Rule: A move from (x, y) takes you to (y, x).

Consider the set whose elements are the points

$$(2, 1), (4, 0), (6, -1).$$

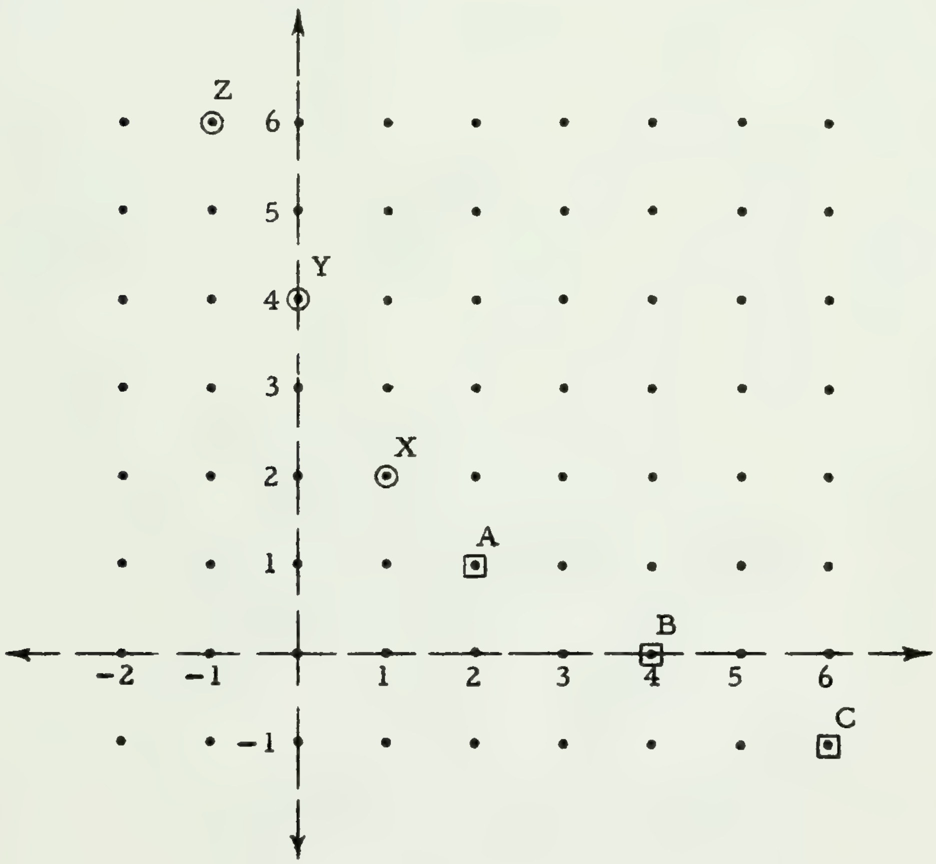
Make one move from each of these points and describe the new set of points.

Solution. From A, $(2, 1)$, move to X, $(1, 2)$.

From B, $(4, 0)$, move to Y, $(0, 4)$.

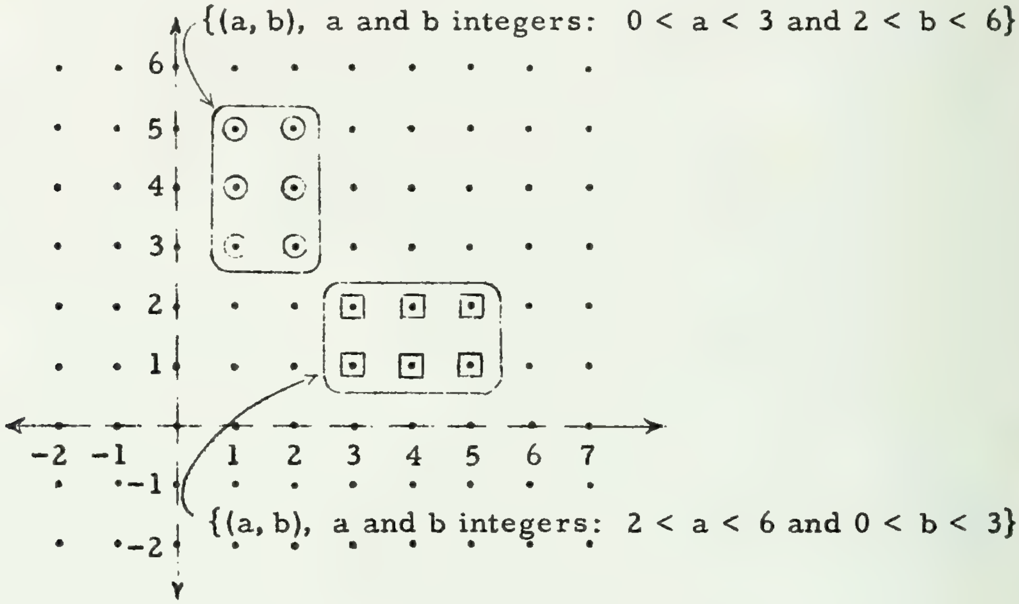
From C, $(6, -1)$, move to Z, $(-1, 6)$.

The new set is $\{(1, 2), (0, 4), (-1, 6)\}$.



Sample 3. Rule: A move from (x, y) takes you to (y, x).
Consider the points in
$\{(a, b), \text{ a and b integers: } 2 < a < 6 \text{ and } 0 < b < 3\}$.
Describe the set obtained by making one move from each point of the given set.

Solution. The points in the given set are shown by boxed dots, and the points in the new set are shown by circled dots.



1. Rule: A move from (x, y) takes you to (x, y - 2). Start at (0, 4) and make 3 moves. What is the final point?

2. Rule: A move from (x, y) takes you to (x + 2, y - 3). Start at (3, 3) and make 3 moves. What is the final point?

[Note: In the following exercises we shall give an abbreviated form for each rule. For example, the statement of the rule in Exercise 2 could have been abbreviated as:

$$(x, y) \rightarrow (x + 2, y - 3).]$$

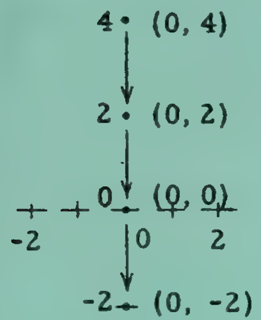
3. Rule: $(x, y) \rightarrow (2x, 2y)$.
Start at (1, 2) and make 5 moves. What is the final point?

4. Rule: $(x, y) \rightarrow (3x, 2y)$.
Start at (0, 0) and make 10 moves. What is the final point?

?

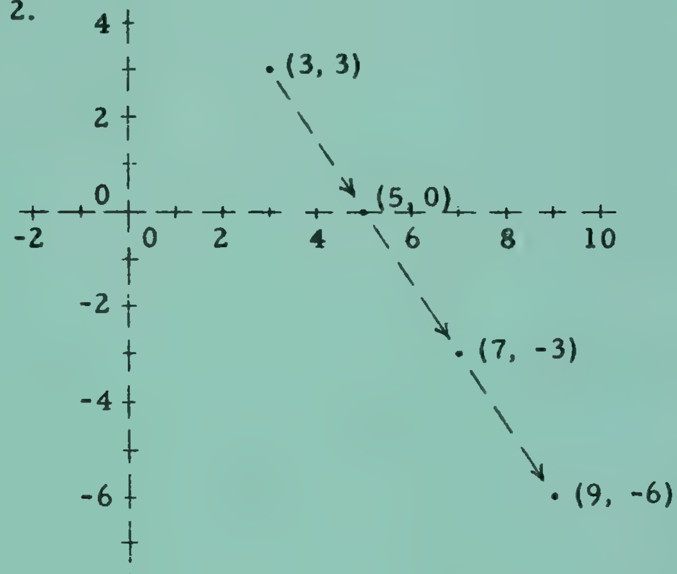
Answers for Part B [on pages 4-18 and 4-19].

1.



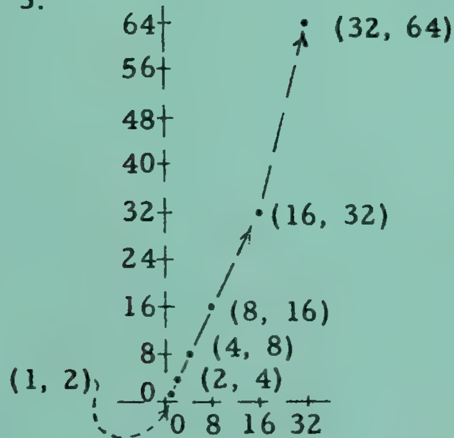
Final point: (0, -2)

2.



Final point: (9, -6)

3.



Final point: (32, 64)

4. No picture is necessary here, of course! Final point: (0, 0)

Sa

[4-19]

?

Review Quiz.

Find the elements in each set.

1. $\{\square: \ 11\square + 3 = 135\}$

2. $\{w: \ w^2 - 10w = 96\}$

3. $\{r: \ \frac{1}{2}r - 13 = 65\}$

4. $\{m: \ \frac{m}{12} + 18 = 126\}$

5. $\{\Delta: \ -\frac{1}{3}\Delta + 10 = -\frac{3\Delta}{4} - 8 + \frac{5\Delta}{12}\}$

6. $\{h: \ 2.5h + 45 = \frac{5h + 92}{2} - 1\}$

7. $\{d: \ \frac{4}{3d - 8} = \frac{2}{5d + 3}\}$

8. $\{z: \ \frac{3z - 4.5}{z} = 0\}$

9. Dave is now two years older than his brother Joe. In eight years, the ratio of their ages will be 6 to 5. What is the present age of each?

10. Gene can complete a certain job in 5 days. When Dave works with him, the job can be done in 3 days. How long would it take Dave to do the job if he worked alone?

*

Answers for Quiz.

1. 12	2. 16, -6	3. 156	4. 1296
5. [none]	6. [Each real number belongs to the set.]		
7. -2	8. 1.5	9. Dave, 4; Joe, 2	
10. $7\frac{1}{2}$ days			

[4-19]

?

the components is 25]. Since the moving rule is:

$$(x, y) \rightarrow (x + 2, y - 3),$$

a point in the new set has a first component which is 4 more than the first component of the corresponding point in the old set; the second component of the new point is 6 less than the second component of the corresponding old point. So, for each point in the new set, if you subtract 4 from its first component, and add 6 to its second, you get the components of a point in the old set. Thus,

if (m, n) belongs to the new set

then (m - 4, n + 6) belongs to the old set.

But, since (m - 4, n + 6) belongs to the old set, it follows that

$$(m - 4)^2 + (n + 6)^2 = 25.$$

So, for each point (m, n) in the new set, $(m - 4)^2 + (n + 6)^2 = 25$. A descriptive name for the new set is:

$$\{(m, n), \text{ m and n integers: } (m - 4)^2 + (n + 6)^2 = 25\}.$$

[You can, of course again use 'c' and 'd':

$$\{(c, d), \text{ c and d integers: } (c - 4)^2 + (d + 6)^2 = 25\}.]$$

In most of these exercises the student has been concerned with what happens to a single point or to a few points. However, one could ask about what happens to <u>every</u> point in the lattice. Some "moves" could be interpreted as a sliding or turning of the entire plane. [Think of a piece of transparent lattice paper sliding above another.] With this interpretation, our moves are transformations of the plane into itself.

*

Here are two other games suggested by students in one of our pilot schools. They imagined a grasshopper making these "jumps".

(1) $(x, y) \rightarrow (2x - 2, 4 - 2y)$

After making 4 jumps, the grasshopper landed on (34, −100). Where did he start?

(2) $(u, v) \rightarrow (2u + 5, 3 - 4v)$

After making 3 jumps, the grasshopper landed on (43, 231). Where did he start?

*

[4-19]

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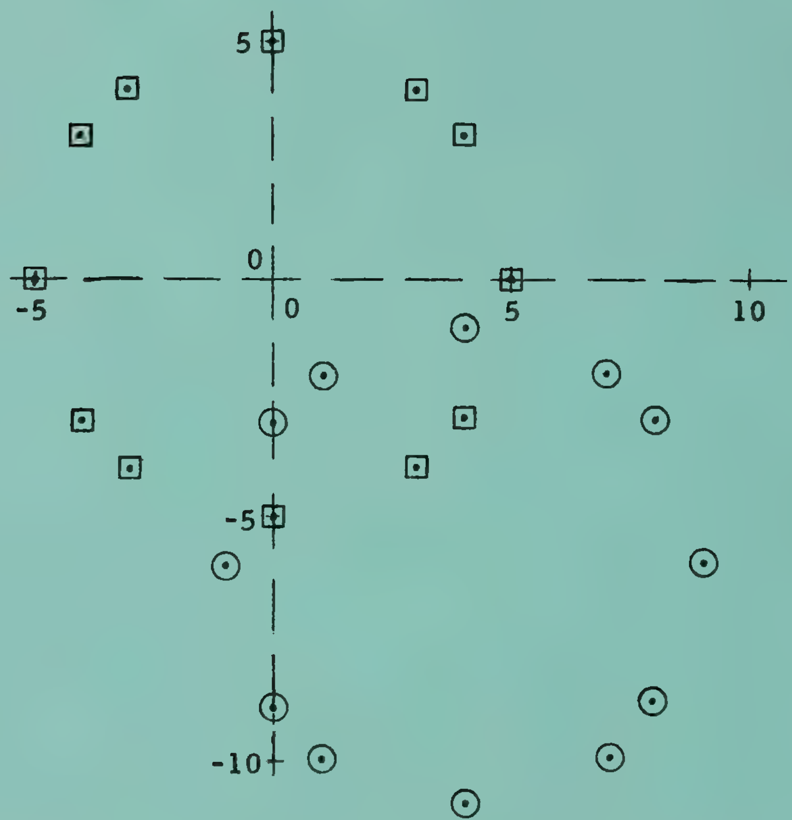
13.



Students usually find this exercise quite fascinating.

The given set contains twelve points arranged on the circle with center at (0, 0) and radius 5. The two moves "translate" the points in the given set to a circle with radius 5 and center at (4, −6). [Ask the students whether they can determine the center of the circle to which the points of the new set belong.]

A simple way of naming the new set is to list its elements.

$\{(4, -1), (9, -6), (4, -11), (-1, -6), (7, -2), (7, -10),$
$(1, -2), (1, -10), (8, -3), (0, -3), (8, -9), (0, -9)\}$

But, some students will want to use a set selector to construct a name. Let's consider a point in the new set. It was obtained by a double move of a point in the old set, [and the components of this point in the old set are such that the sum of the squares of

[4-19]

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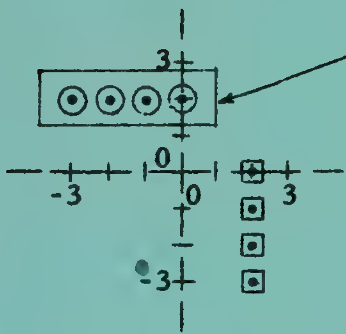
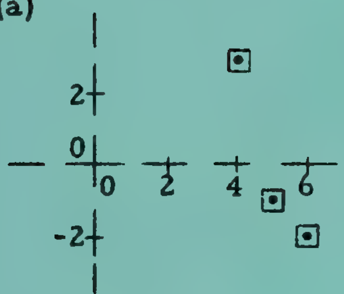
10. 

$\{(0, 2), (-1, 2), (-2, 2), (-3, 2)\}$ is the new set.

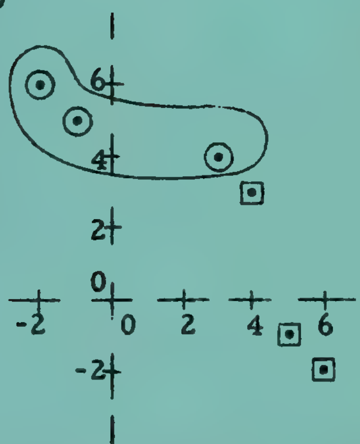
[Ask the class to use brace-notation to describe this new set (without listing its elements). Here is such a description:

$\{(x, y)$, x and y integers: $y = 2$ and $0 \geq x \geq -3\}$.]

11. (a) 

The new set is the same as the given set. [You may want to ask your class whether they can use brace-notation to describe this set (without listing its elements). Here is such a description:

$\{(x, y)$, x and y integers: $(x = 4$ and $y = 3)$ or $(x = 5$ and $y = -1)$ or $(x = 6$ and $y = -2)\}$.]

(b) 

$\{(3, 4), (-1, 5), (-2, 6)\}$ is the new set.

Here is another description of the new set:

$\{(x, y)$, x and y integers: $(x = 3$ and $y = 4)$ or $(x = -1$ and $y = 5)$ or $(x = -2$ and $y = 6)\}$.

(c) Resulting set is same set as starting set.

(d) Resulting set is same set as starting set.

(e) Resulting set is same set as the one obtained in (b).

12. The given set is the number plane lattice itself. Each point in this set is "jumped" to another point in the set because the sum of an integer and 1 is an integer. Hence, the new set is equal to the given set, and we could use the same name for the new set as we did for the given set.

[4-19]

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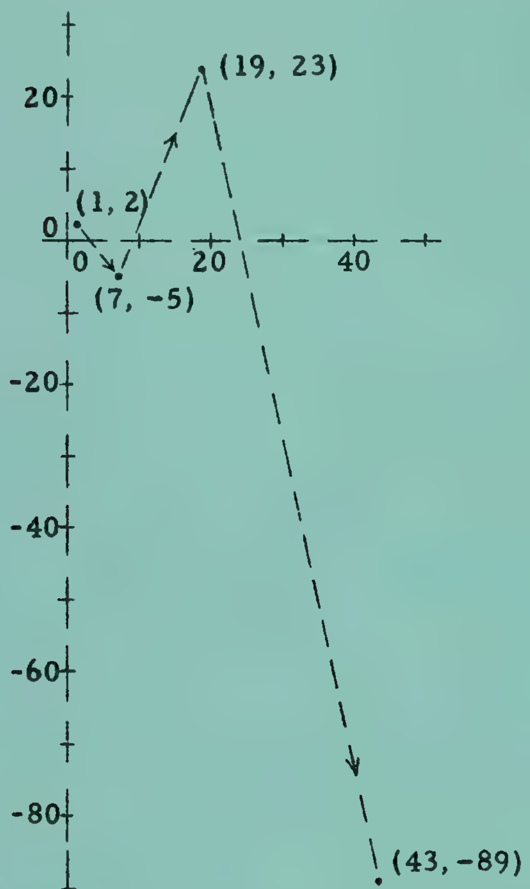
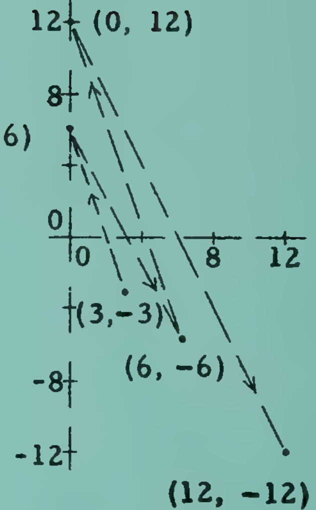
Here is a picture showing the moves.

7. Starting point: (1, 2)



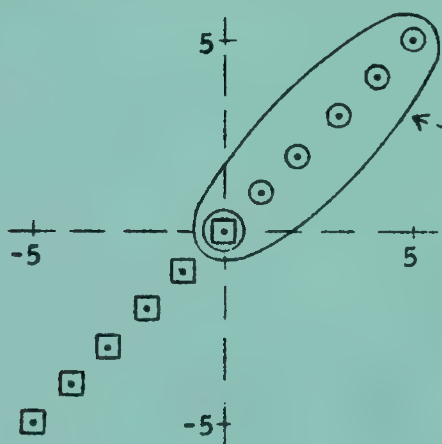
8.



Final point: (12, -12)

[In Exercises 9, 10, 11, and 13 we use boxed dots to show the points in the given set, and circled dots to show the points in the new set.]

9.



{(a, b), a and b integers: $a = b$ and $0 \leq a \leq 5$} is the new set.

[4-19]

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A bit more is involved with Exercise 7. First, consider the first component of the final point, 43. What does the rule tell us about how this point was obtained? The students will probably say that "it tells us that 5 was added to some number to get 43". Then ask, "Is that all the rule tells us?" Students should be able to tell that the number obtained by subtracting 5 is twice the value of 'u', and that the value of 'u' will be the first component of the point obtained at the end of the second move. Write on the board:

$$43 - 5 = 38, \quad 38 = 2 \times 19.$$

So, 19 must be the first component of the point obtained at the end of the second move.

Start the same kind of discussion in regard to the second component of the final point, -89. We expect that soon some student will say, "Couldn't we figure these out by setting up an equation? Why not write '$3 - 4v = -89$' and then solve? Won't that get us the second component of the point obtained at the end of the second move?" When this happens, carry out the student's suggestion at the board, like this.

Final point: $(43, -89)$

First Component	Second Component
$2u + 5 = 43$	$3 - 4v = -89$
$u = 19$	$v = 23$

At the end of the second move, the point obtained was $(19, 23)$.

To get the point obtained at the end of the first move, again write the equations. [Ask the class what they should be.]

$2u + 5 = 19$	$3 - 4v = 23$
$u = 7$	$v = -5$

At the end of the first move, the point obtained was $(7, -5)$.

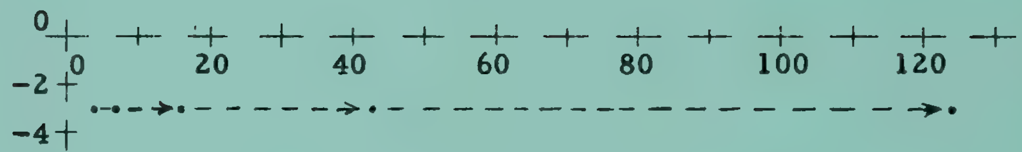
Now, to get the starting point, we need one more pair of equations. Your class should tell you these.

$2u + 5 = 7$	$3 - 4v = -5$
$u = 1$	$v = 2$

The starting point was $(1, 2)$.

[4-19]

5.



Final point: (124, −3)

*

Exercises 6 and 7 should be discussed in class. For Exercise 6, ask questions like this to help the class get started:

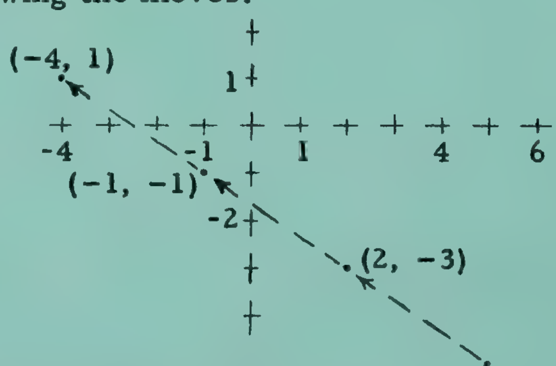
(a) We know that the first component of the final point is −4, and the rule tells us that 3 was subtracted from some number to obtain −4; what was this number? Will this number be the first component of the point obtained immediately before the final point?

(b) We know that the second component of the final point is 1, and that 2 was added to some number to obtain 1; what was this number? Is this number the second component of the point obtained immediately before the final point?

After the class has determined that the point reached by the second move was (−1, −1), go through a sequence of questions like (a) and (b) to help them ascertain the point reached by the first move [(2, −3)]. Finally, by thinking through the same kind of questions, they can find that the starting point was (5, −5).

Here is a picture showing the moves.

6.



Starting point: (5, −5)

5. Rule: $(j, k) \rightarrow (3j - 5, 2k + 3)$.
Start at $(4, -3)$ and make 4 moves. What is the final point?

6. Rule: $(x, y) \rightarrow (x - 3, y + 2)$.
After making 3 moves, the final point is $(-4, 1)$. Give the starting point.

7. Rule: $(u, v) \rightarrow (2u + 5, 3 - 4v)$.
After making 3 moves, the final point is $(43, -89)$. Give the starting point.

8. Rule: $(x, y) \rightarrow (x + y, x - y)$.
Start at $(3, -3)$ and make 4 moves. What is the final point?

9. Rule: $(s, t) \rightarrow (|s|, |t|)$.
Make one move from each point in

$$\{(a, b), \text{ a and b integers: } a = b \text{ and } -5 \leq a \leq 0\},$$

and describe the set of points which you obtain.

10. Rule: $(x, y) \rightarrow (y, x)$.
Make one move from each point in $\{(2, 0), (2, -1), (2, -2), (2, -3)\}$, and describe the new set of points.

11. Rule: $(x, y) \rightarrow (y, x)$.
   (a) Start at each point in $\{(4, 3), (5, -1), (6, -2)\}$, and make two moves. Describe the resulting set.
   (b) Repeat (a) but, starting at each point in the given set, make three moves.
   (c) Repeat (a) but with four moves.
   (d) Repeat (a) but with any even number of moves.
   (e) Repeat (a) but with any odd number of moves.

12. Rule: $(y, x) \rightarrow (y + 1, x + 1)$.
Make one move from each point in

$$\{(m, n), \text{ m and n integers: } m + n = n + m\},$$

and describe the resulting set.

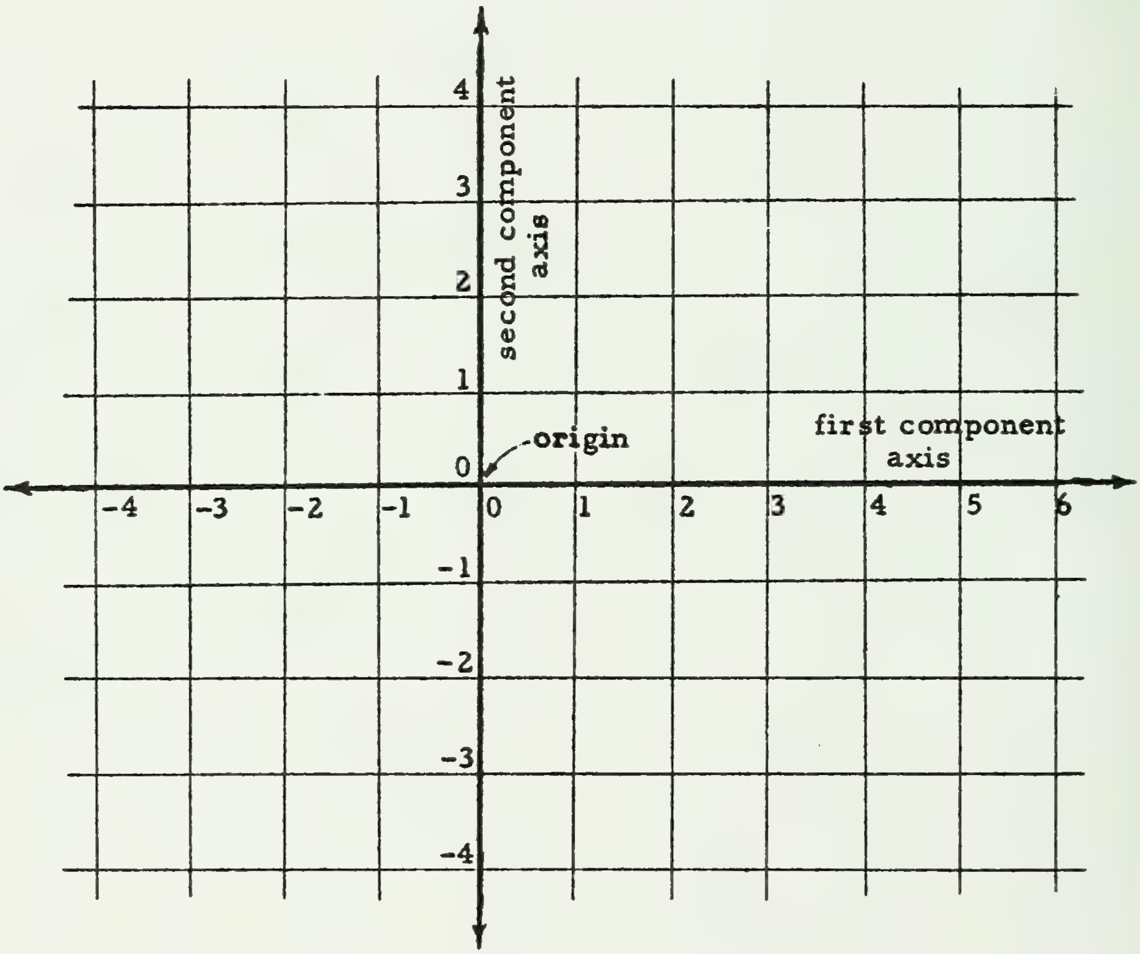
13. Rule: $(x, y) \rightarrow (x + 2, y - 3)$.
Start at each point in

$$\{(c, d), \text{ c and d integers: } c^2 + d^2 = 25\}$$

make two moves, and describe the resulting set.

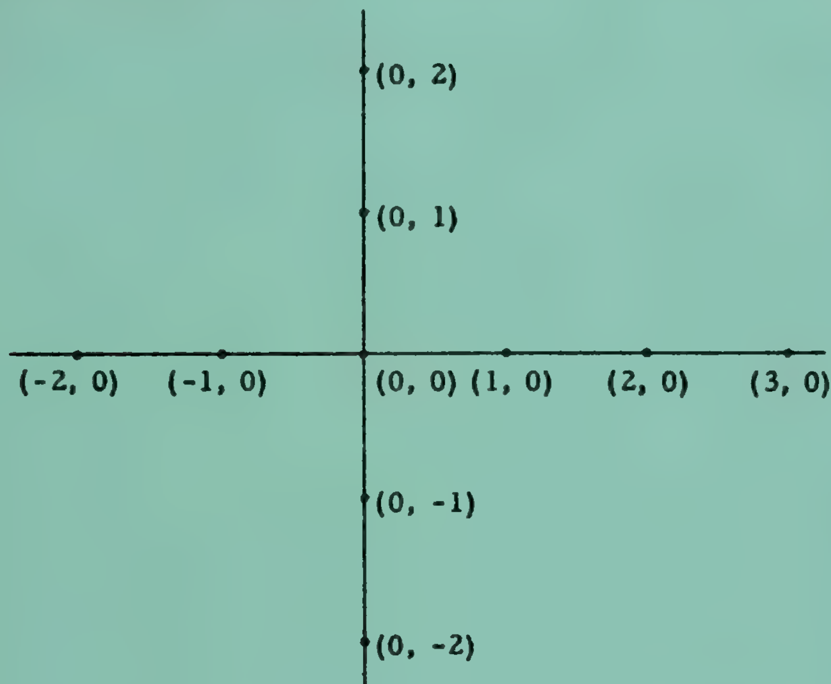
4.02 <u>The number plane</u>.--Up to now you have been working with the number plane lattice--that is, the cartesian square of the set of <u>integral</u> real numbers. Now we shall consider the cartesian square of the set of <u>all</u> real numbers. This cartesian square is called <u>the number plane</u>. [Do you see that the number plane lattice is a subset of the number plane?]

As in the case of the number line, you can draw a picture of only a part of the number plane. The usual procedure is illustrated here.



Notice that the pictures of the component axes are made more prominent than the other columns and rows in the picture. These other lines are called <u>grid lines</u>. A vertical grid line pictures a set of ordered pairs of real numbers which have the same first component, and a horizontal grid line pictures a set of ordered pairs which have the

Note that the component axes of the number plane are sets of ordered pairs of real numbers. In particular, the first component axis is the set of ordered pairs of real numbers with second component 0. The first component axis is not the number line [the number line is a set of real numbers]. To emphasize this point you should make a picture of the number plane like this:



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The question about the correspondence between dots on the picture and ordered pairs of real numbers contains a subtle difficulty of which you should be aware. Once you have drawn a picture of the number plane, you can draw a dot corresponding with any ordered pair of real numbers whose components are such that the dot falls within the boundaries of the picture. But, for a given picture no matter what the scale, there are bound to be two ordered pairs for which you cannot make different dots. It is a fact of mathematics and a fact of physics that there are many more ordered pairs of real numbers than there are possible dots.

4.0?

r

same second component. Just as in the case of the number line, we must frequently make estimates when plotting points. For example, the ordered pair (2.5, 3.1) corresponds with a dot which is midway between the 2-column and the 3-column, and one-tenth of the way from the 3-row to the 4-row. So, we would have to make estimates to mark this dot on the picture.

An ordered pair of <u>real</u> <u>numbers</u> is a point of the number plane [you recall that an ordered pair of integers is a point of the number plane lattice], and corresponds with a dot which can be marked on the picture. The dot is the graph of the ordered pair, and the components of the ordered pair are the coordinates of the dot. The first coordinate of a dot is sometimes called its <u>abscissa</u>, and the second coordinate is sometimes called its <u>ordinate</u>.

Study the diagram on page 4-20 and follow these instructions.

(1) Point to the grid line which contains dots with abscissa (first coordinate) 3.

(2) Point to the grid line which contains dots with ordinate 2.

(3) Point to the grid line which contains dots with ordinate 4.

(4) Point to the grid line which contains dots with abscissa -1.

(5) Point to the grid line which contains dots with ordinate 0.

(6) Draw the grid line which contains dots with abscissa $\frac{1}{2}$.

(7) Draw the grid line which contains dots with ordinate $-2\frac{1}{3}$.

(8) Point to the grid lines which contain the graph of (5, 3). Of (2, -1). Of (-3, -4). Of (4, 4). Of ($\frac{1}{2}$, 2). Of (0, 0).

(9) Draw the grid lines which contain the graph of ($3\frac{1}{2}$, $2\frac{1}{2}$). Of (-2.5, 1.5). Of (-3.5, -3.5).

In making a picture of part of the number plane you are completely free in your choice of which grid lines to draw. [It is customary, however, to include pictures of the component axes.] Since the only purpose of the grid lines is to help you plot points, the components of the points to be plotted in a particular problem determine the selection of grid

lines. For example, the diagram on page 4-20 would be completely useless if you wanted to plot the graph of, say, (10, 17). When you use cross section paper [or graph paper, as it is commonly called], you will find equally spaced grid lines already printed on the paper. Choose two of these grid lines to represent the component axes and use a pencil to make them more prominent. Then decide upon the numbers to be assigned to each grid line. You do this by selecting the dots which are to correspond with (1, 0) and (0, 1). In other words, you select a scale for each axis. Here are several examples. Note that the scales differ from diagram to diagram even though the smallest distance between parallel grid lines is the same for all of the diagrams.

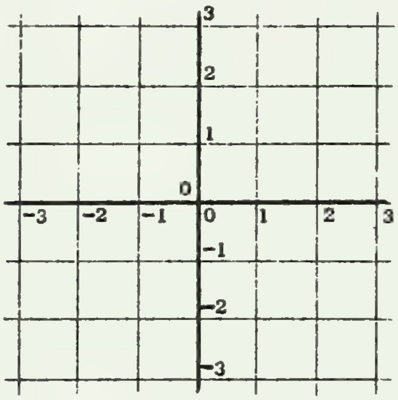


Figure 1.

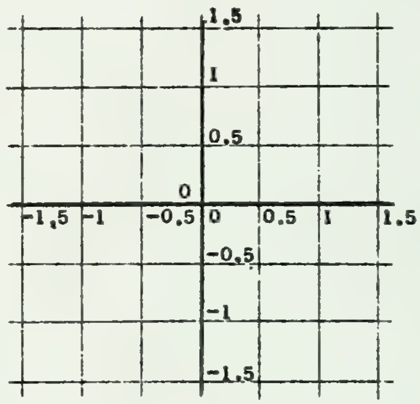


Figure 2.

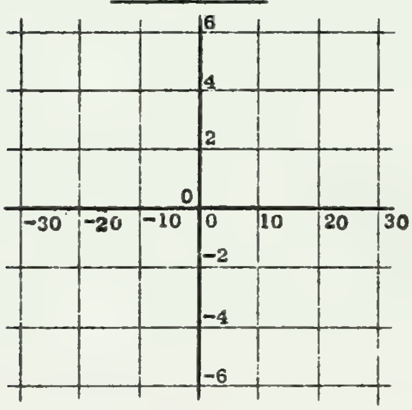


Figure 3.

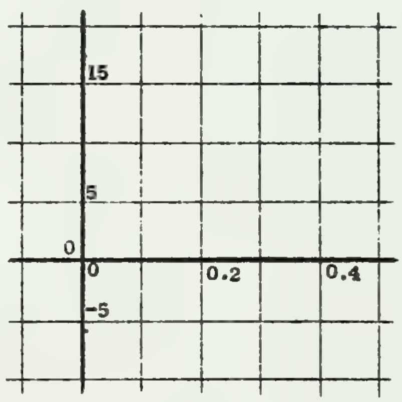


Figure 4.

In Figure 1 the same scale is used on both axes. The same scale is used on both axes in Figure 2. The scales for the axes differ in Figure 3 and 4. Note [Figure 4] that it is not necessary to label each grid line. When the grid lines are equally spaced, you can always tell the numbers to be assigned to "in-between" grid lines.

[4-23]

line

of region D which is to the left of the second component axis, the arrow pointing to the origin should point down, and slightly to the right.

One of our pilot school teachers reported that his class found it interesting to imagine that one wall of the classroom was the "chart" on which these regions were to be pictured. They located the axes on the wall (by pointing), and considered what scale might be used so that the pictures of the regions would lie on the wall of the classroom.

*

Quiz.

Simplify.

1. $3(m - 3n) + 2(4n - m)$
2. $-8(a + 5c) - (6a - 7c)$
3. $(4\square - 5\Delta)6 - 7(\Delta + 3\square)$
4. $\frac{1}{3}(15x + 12y) - \frac{1}{5}(5x - 15y)$
5. $\frac{-9e + 27f}{3} + \frac{-9e + 36f}{9}$
6. $(14.7r - 29.4s) \div 7 + (3.2s - 5.6r) \div (-8)$

*

7. Which of the following equations has more than one root?

   (a) $|13 - 8x| + 32 = 7$ (b) $|9 + m| = 0$

   (c) $|4r - 3| + 7 = 2 + |3 - 4r|$

   (d) $|5 + 3x| + 9 = 45$ (e) $|a + 5.2| + 7.2 = 3$

8. A contractor employed 150 men with a daily payroll of \$2870. The unskilled laborers earned \$15 per day and the skilled laborers \$25 per day. How many of each did he employ?

9. How many ordered pairs of numbers are there with first components chosen from $\{3, 5, 7\}$ and with second components chosen from $\{-2, -4, -6, -8\}$? List the ordered pairs.

*

Answers for Quiz.

1. $m - n$ 2. $-14a - 33c$ 3. $3\square - 37\Delta$ 4. $4x + 7y$

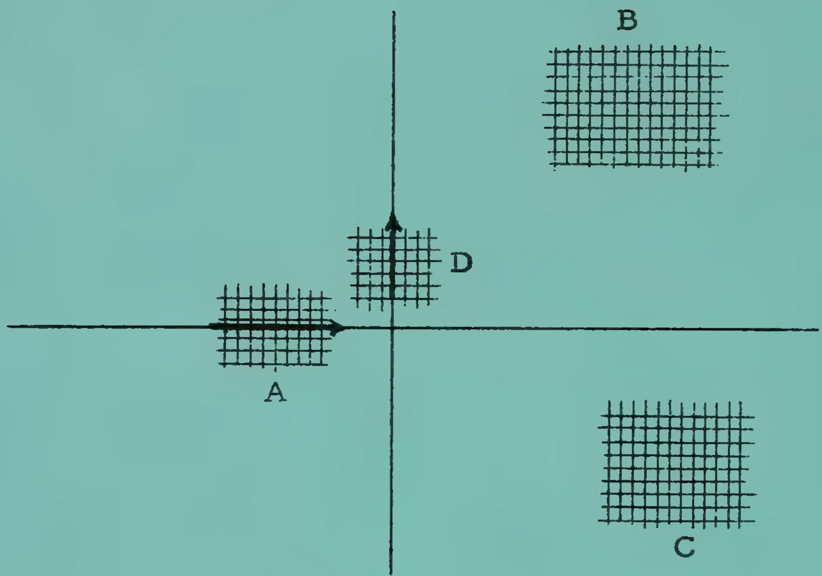
5. $-4e + 13f$ 6. $2.8r - 4.6s$ 7. (d)

8. unskilled, 88; skilled, 62 9. 12

TC[4-23, 24]c

[4-23]

2.



3. (a) yes; (b) yes

4. The direction in which the arrow points will depend, of course, on where one assumes the stranger to be. If one meets him in the portion of region A which is "above" the first component axis, the arrow pointing to the origin should point to the right and be tilted downward slightly. If one meets the stranger along the first component axis [the arrow on the picture of region A], the arrow drawn to point to the origin will coincide with the one given on the picture. Finally, if one meets the stranger in the portion of region A which is below the first component axis, the arrow pointing to the origin should point to the right and be tilted upward slightly.

5. For region B, the arrow should point toward the left, and be tilted downward at about a 45°-angle.

   For region C, the arrow should point to the left, and be tilted upward slightly.

   For region D, if one assumes that he meets the stranger to the right of the second component axis [the arrow on the picture of region D], the arrow should point down, and slightly to the left. If one meets the stranger along the second component axis, the arrow drawn to point to the origin will coincide with the one given in the picture. Finally, if one meets the stranger in the portion

[4-23]

We suggest that the exercises in Part A be done in class. You might work out Exercise 1 together. Then, have the remaining four exercises worked by the students individually. [It will be helpful if the students have graph paper available!] You can move among the students to see that they are choosing a convenient scale for the axes.

*

Part B drives home the point that it is only convention which makes ordinary pictures of the number plane always contain the graph of (0, 0).

You should spend 10 or 15 minutes discussing the graph paper following page 4-131. The horizontal arrow shows the first component axis, and points toward the graph of the origin. The vertical arrow shows the second component axis and points away from the graph of the origin.

We intentionally included some points which are not in any of the regions. [The commas have been omitted from the numerals in these exercises because they could be confused with the commas that separate the names for the two components of the ordered pair.]

*

Answers for Part B. [We give the name of the region in which the point is located.]

1. H...Region B   J...Region A   K...Region D
   L...Region C   M...Region B   N...Region A
   O...Region D

   Points P, Q, and R are not in any of the regions. [Ask students to compare point Q with point K.]

EXERCISES

A. For each exercise, draw a picture of part of the number plane and graph the ordered pairs given. Select scales so that all the pairs in the exercise can be conveniently plotted on the same picture.

1. (3, 5), (2, −1), (4, −3), (0, −2), (−3, 4)
2. (2.5, 3.5), (−1.5, 1.5), (0, −2.5), (−3.5, 1.5), (3, −2)
3. (20, 3), (−40, 5), (30, −4), (−50, 1), (0, −2)
4. (1, 1), (−1, −1), (2, 8), (−3, −27), (4, 64)
5. (0.3, 7), (−0.2, −8), (−1, 3), (−0.8, 0), (0.5, −5)

B. At the end of this unit you will find several sheets of paper showing four regions of the number plane. Use two sides of one sheet for the five exercises below and on the next page. [You will need the rest of the sheets for later exercises.]

1. Graph each of these ordered pairs, and label the graph with the appropriate letter.

H: (1 000 003, 1 000 003)	J: (−156 616, −2)
K: (5, 135 410)	L: (999 999, −450 003)
M: (1 000 008, 999 998)	N: (−156 620, 0)
O: (0, 135 406)	P: (156 602, 4)
Q: (135 411, −2)	R: (0, 0)

2. Draw a picture of part of the number plane [include the two component axes] and make a rough sketch of the location of the four regions.

(continued on next page)

3. (a) Is there a straight line in the number plane which intersects in a nonempty set each of the regions A and B?

   (b) Is there a straight line in the number plane which intersects in a nonempty set each of the regions C and D?

4. Suppose you were in Region A and you met a stranger who asked the direction in which he should walk to get to the origin. Indicate (by drawing an arrow on the picture for Region A) which way you would point to show him the location of the origin.

5. Repeat Exercise 4 for each of the other three regions.

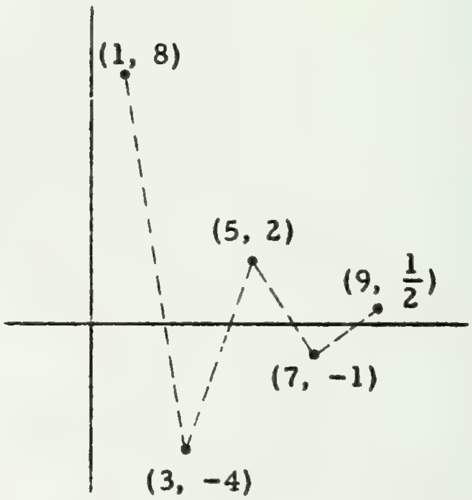
C. Number Plane Games.

Use a picture of the number plane to keep a "running" record of the moves.

Sample. Rule: $(x, y) \rightarrow (x + 2, -\frac{1}{2}y)$.
Start with (1, 8) and make 4 moves. What is the final point?

Solution. $(1, 8) \rightarrow (3, -4)$
$(3, -4) \rightarrow (5, 2)$
$(5, 2) \rightarrow (7, -1)$
$(7, -1) \rightarrow (9, \frac{1}{2})$

The final point is $(9, \frac{1}{2})$.



1. Rule: $(x, y) \rightarrow (x + 3, \frac{1}{2}y)$.
   Start at $(-5, -8)$ and make 4 moves.
   What is the final point?

[4-25]

?

9. (a) $(2, 3) \leftarrow (4, 9) \leftarrow (16, 81) \leftarrow (256, 6561)$

starting point

(b) The game cannot start at (16, −9) because there is no non-negative real number whose square is −9. ['$\sqrt{-9}$' is nonsense.] Of course, you can start at (16, −9) but you can't play the game.

(c) You can't, because there is no real number whose principal square root is negative. [Of course, if you start at (−2, −3), there are some who would argue that this is where you would be after 2 moves. But, it is better to say that a starting point is one from which you can make a move, even if the move takes you back to the starting point.]

10. (a) $(3, 5) \rightarrow (\frac{1}{3}, \frac{1}{5}) \rightarrow (3, 5) \rightarrow (\frac{1}{3}, \frac{1}{5})$

final point

(b) (3, 5) is the final point.

(c) $(\frac{1}{3}, \frac{1}{5})$ is the final point.

(d) You can't play the game if you start at (2, 0) because you can't make a move. [0 has no reciprocal.]

(e) You can't play the game if you want to end at (0, 0) because there is no number whose reciprocal is 0.

[4-25]

?

[We are omitting pictures for the remainder of the exercises in Part C, since it would take so much space to include them. Some of the components of the points involved in these games are rather large, so it would be difficult to choose a convenient scale. You may want to relax the requirement that the students draw pictures for the last 5 or 6 exercises.]

5. $(-9, -6) \leftarrow (-5, -2) \leftarrow (-3, 2) \leftarrow (-2, 6)$

$$\underset{\text{starting point}}{\uparrow}$$

[For an explanation of how one determines the starting point, see TC[4-19]a and [4-19]b.]

6. $(109, -9) \leftarrow (37, -5) \leftarrow (13, -3) \leftarrow (5, -2)$

$$\underset{\text{starting point}}{\uparrow}$$

7. (a) 6 moves

   (b) It can't be done. You can move as close to the origin as you please, but you can never get there. [What is the smallest number of moves to get you closer to (0, 0) than (1/10, 1/10)? Answer: 4 moves. What is the smallest number to get you closer than (1/100, 1/100)? Answer: 7 moves. What is the smallest number to get you closer than (1/1000, 1/1000)? Answer: 10 moves.]

8. $(1, 2) \rightarrow (1, 4) \rightarrow (1, 16) \rightarrow (1, 256)$

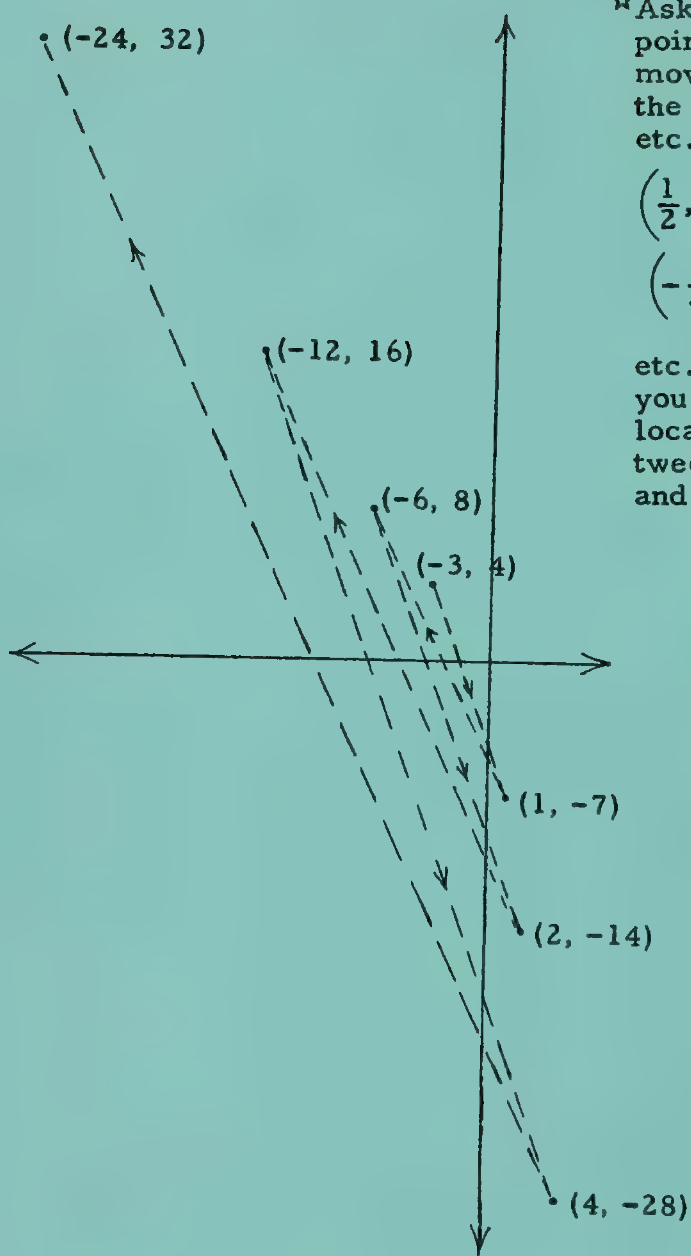
$$\underset{\text{final point}}{\uparrow}$$

   No matter how many moves you make you will always stay on a vertical line and move upward. [Why won't you ever get below the first component axis? Answer: There is no real number whose principal square root is negative.]

[4-25]

?

4.



The final point is $(-24, 32)$.

☆Ask students to guess the point from which you would move to get to $(-3, 4)$. And, the point before that point, etc. [Answers: $\left(\frac{1}{2}, -\frac{7}{2}\right)$, $\left(-\frac{3}{2}, 2\right)$, $\left(\frac{1}{4}, -\frac{7}{4}\right)$, $\left(-\frac{3}{4}, 1\right)$, $\left(\frac{1}{8}, -\frac{7}{8}\right)$, $\left(-\frac{3}{8}, \frac{1}{2}\right)$, etc.] No matter how far back you traced the points, the locations would oscillate between the upper left region and the lower right region.

[4-25]

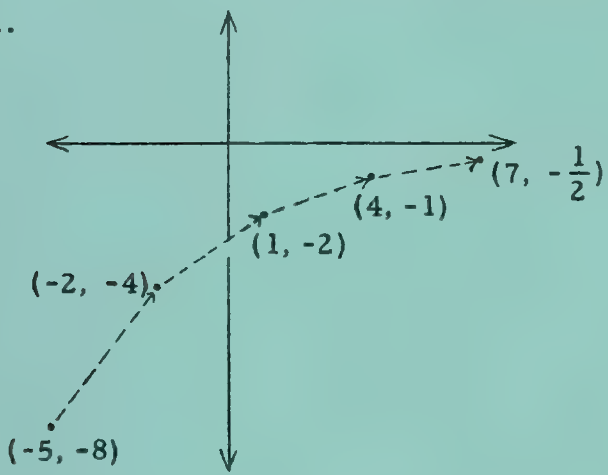
?

3.

Answers for Part C [which begins on page 4-24].

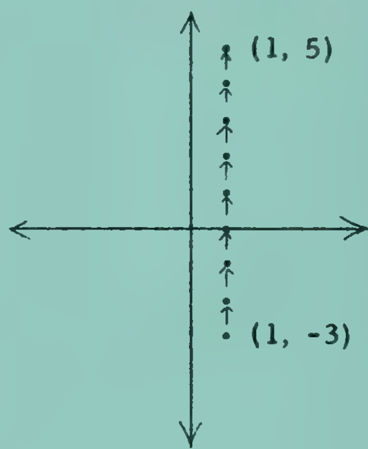
[See the COMMENTARY for Exercise 11 on page 4-89 for a more formal discussion of number plane games.]

1.



The final point is $(7, -\frac{1}{2})$. [No matter how many moves you make, you'll never cross the first component axis.]

2.



The final point is (1, 5). [No matter how many moves you make, you'll never get off the vertical line.]

3. The final point is (4, 2) which is the same as the starting point. Note that 4 is the root of '$3x - 8 = x$' and 2 is the root of '$9y - 16 = y$'. You can think of (4, 2) as "the standstill point" for the given rule. Ask students to tell the standstill points for these rules:

$(x, y) \rightarrow (5x - 8, 7y + 18)$	[(2, −3)]
$(x, y) \rightarrow (8 - x, 3 - y)$	[$(4, \frac{3}{2})$]
$(x, y) \rightarrow (y, x)$	[any point with equal components]
☆$(x, y) \rightarrow (3x + y, 2y - x)$	[(0, 0)]

2. Rule: $(x, y) \to (2x - 1,\ y + 1)$.
   Start at $(1, -3)$ and make 8 moves. What is the final point?

3. Rule: $(x, y) \to (3x - 8, 9y - 16)$.
   Start at $(4, 2)$ and make 20 moves. What is the final point?

4. Rule: $(x, y) \to (x + y,\ x - y)$.
   Start at $(-3, 4)$ and make 6 moves. What is the final point?

5. Rule: $(x, y) \to (2x + 1,\ y - 4)$.
   After 3 moves you are at $(-9, -6)$. What was the starting point?

6. Rule: $(u, v) \to (3u - 2,\ 2v + 1)$.
   After 3 moves you are at $(109, -9)$. What was the starting point?

7. Rule: $(x, y) \to (\frac{1}{2}x,\ \frac{1}{2}y)$.
   How many moves will it take to go from $(64, 64)$ to $(1, 1)$?
   How many moves to go from $(1, 1)$ to $(0, 0)$?

8. Rule: $(x, y) \to (x^2,\ y^2)$.
   Start at $(1, 2)$ and make 3 moves. What is the final point?

9. Rule: $(x, y) \to (\sqrt{x},\ \sqrt{y})$.
   After 3 moves you are at $(2, 3)$. What was the starting point? Can you start this game at $(16, -9)$? Where can you start the game so that after 2 moves you are at $(-2, -3)$?

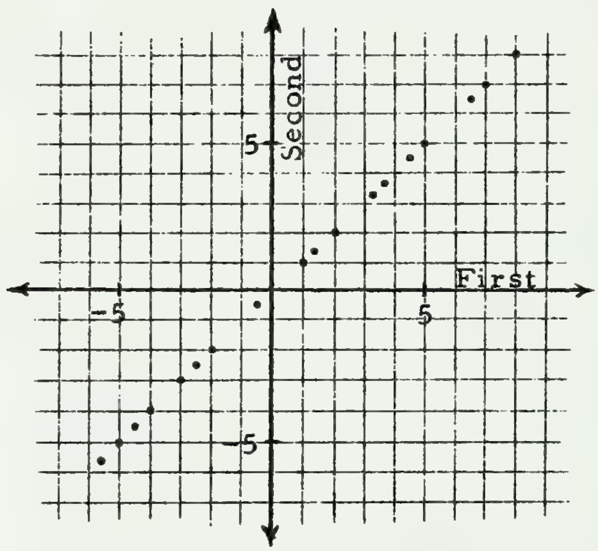
10. Rule: $(x, y) \to (\frac{1}{x},\ \frac{1}{y})$.
    Start at $(3, 5)$ and make 3 moves. What is the final point? Start at $(3, 5)$ and make 1000 moves. What is the final point? What is the final point if you start at $(3, 5)$ and make 1001 moves? Can you use this rule if you start at $(2, 0)$? Where can you start if you want to end at $(0, 0)$?

EXPLORATION EXERCISES

A. For each of the sets of ordered pairs described below, plot as many of the ordered pairs as you can on a picture of the number plane.

Sample. The set of all ordered pairs of real numbers such that the first component is equal to the second component.

Solution. Some of the ordered pairs in this set are (8, 8), (−2, −2), (1.3, 1.3), (π, π), (−5, −5), and (0, 0). Since this is an infinite set, we can't plot all of them. But, we can plot quite a few of them until we see some sort of pattern.



Do you see a pattern? Can you fill in more dots? Do you see a quick way of getting all the dots?

1. The set of all ordered pairs of real numbers such that the first component is equal to the product of −2 by the second component.
2. The set of all ordered pairs of real numbers such that the first component is 2 more than the second component.
3. $\{(x, y): \ x = \frac{1}{2}y + 3\}$
4. $\{(a, b): \ a + b = 9\}$
5. The set of all ordered pairs of real numbers such that the first component is 4
6. $\{(x, y): \ y = 2\}$

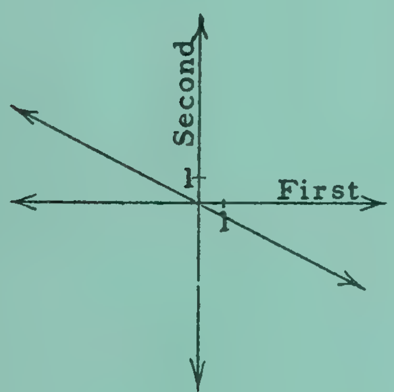
The purpose of the Exploration Exercises is to prepare students for the problem of drawing the locus of a sentence. You might do the Sample and Exercise 7 at the board, and use the rest of Part A as seatwork and homework. Do the Sample of Part B at the board.

*

Answers for Part A [on pages 4-26 and 4-27].

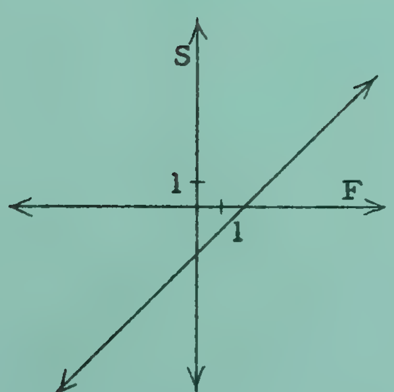
[Be sure students understand that they are to exhibit the patterns in these problems by drawing straight lines, and shading regions. They can indicate the infinite extent of these figures by putting in arrows. You should emphasize these characteristics of the pictures by completing the picture in the Sample.]

1.



To get started on this exercise, a student might say, "Suppose the first component of one of the ordered pairs is 4. What is the second component? −2 times the second component is 4. So, the second component is −2. Hence, (4, −2) is one of the pairs." He should then plot this pair immediately.

2.



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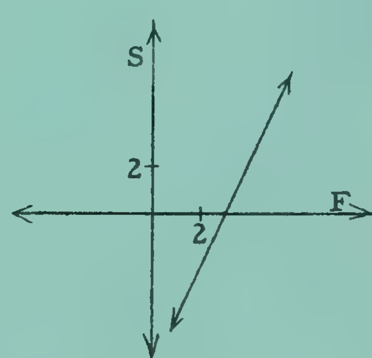
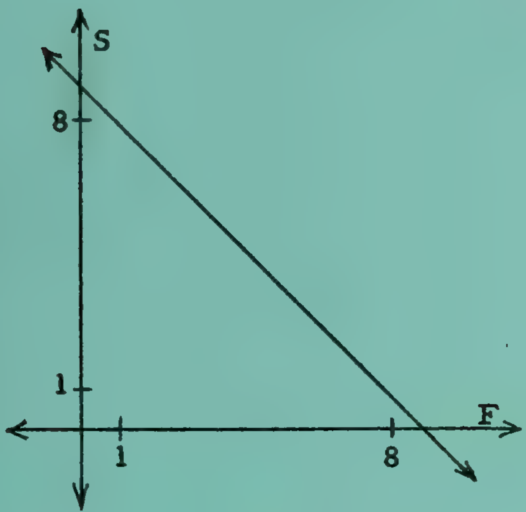


[4-26]

A

4/58

4.



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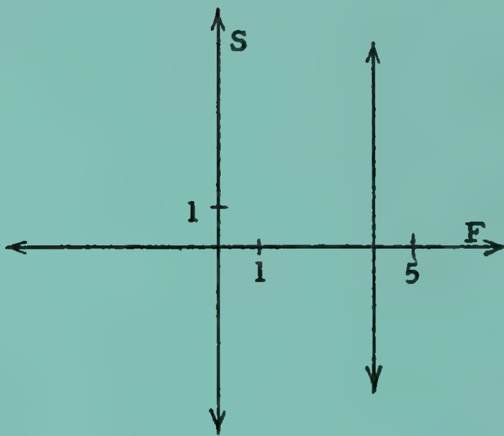

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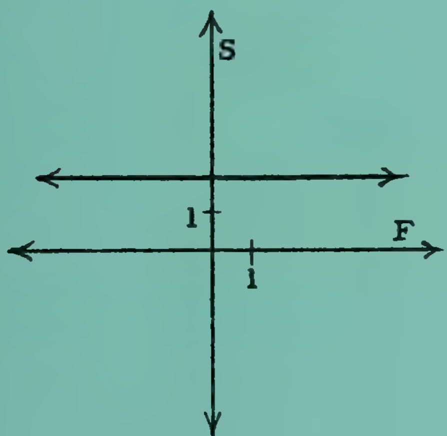

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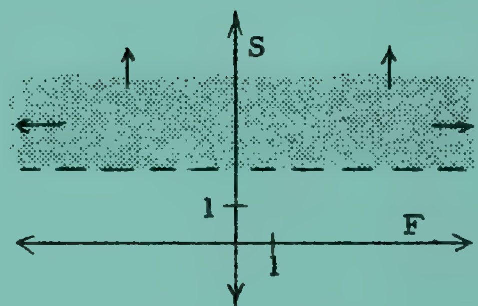
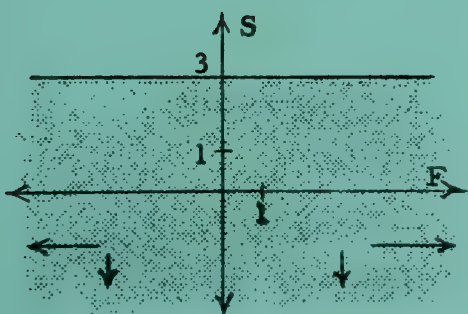
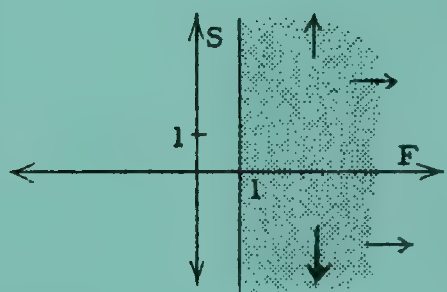


[Note the convention of using a dashed line to show that the boundary is not included.]

8.



9.



[4-26]

A

[4-27]

Review Quiz.

Write an equivalent expression which does not contain parentheses or other symbols of grouping.

1. $5(2a - 7b)$
2. $(x + 8y) \times (-3)$
3. $\frac{1}{3}(12r - 3s)$
4. $(-21m + 30x) \div 3$
5. $-\square(4\Delta - 6\bigcirc)$
6. $(3.5u - 1.2v) \times (-6)$
7. $5ab(-3a - 11c)$
8. $(18e - 39f) \div (-3)$
9. $\frac{9m + 3n}{-6}$
10. $\frac{1}{8}(-32r - 16s)$
11. The expression '$F = \frac{9C}{5} + 32$' is used in converting from degrees C to degrees F. What is the temperature in degrees F corresponding to 22 degrees C?
12. $\{x: -3 < x < 3\}$ is a subset of which of these sets?

    (a) $\{x: -2 < x < 2\}$ (b) $\{x: |x| < 4\}$

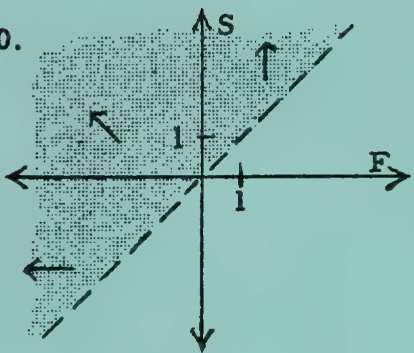
    (c) $\{x: -3 \geq x \geq 3\}$ (d) $\{x: |x| \geq 3\}$

*

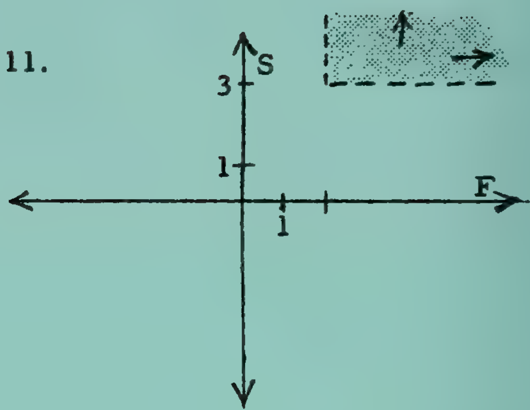
Answers for Quiz.

1. $10a - 35b$
2. $-3x - 24y$
3. $4r - s$
4. $-7m + 10x$
5. $-4\square\Delta + 6\square\bigcirc$
6. $-21u + 7.2v$
7. $15a^2b - 55abc$
8. $-6e + 13f$
9. $-(3m + n)/2$ [or: $-1.5m - .5n$]
10. $-4r - 2s$
11. 71.6
12. (b)

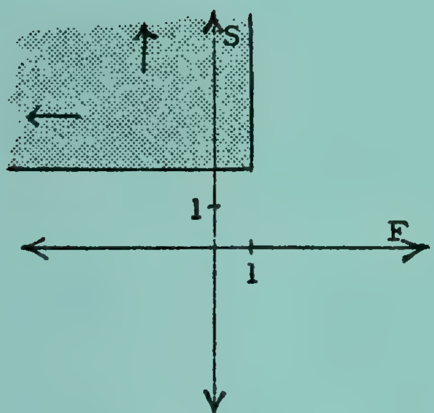
10.



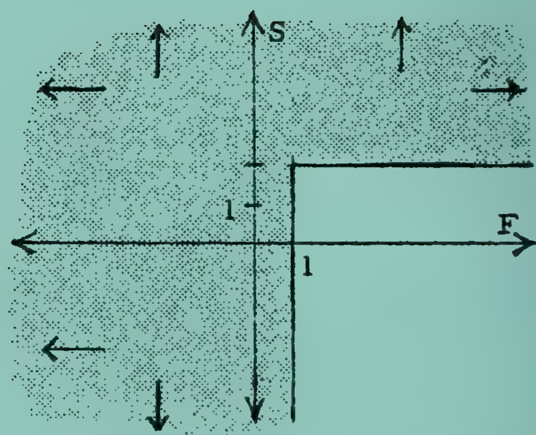
11.



12.



13.



*

Answers for Part B [on pages 4-27 and 4-28].

1. $\{(x, y): \; y \geq 4\}$

2. $\{(x, y): \; y > 4\}$

3. $\{(x, y): \; x \geq -10\}$

4. $\{(x, y): \; x < -\frac{9}{2} \text{ or } y < -3\}$

5. $\{(x, y): \; x < -4 \text{ and } y > 6\}$

6. $\{(x, y): \; x = 3y\}$

*

7. $\{(x, y): y > 2\}$

8. $\{(x, y): y \le 3\}$

9. $\{(x, y): x \ge 1\}$

10. $\{(p, q): q > p\}$

11. $\{(q, p): q > 2 \text{ and } p > 3\}$

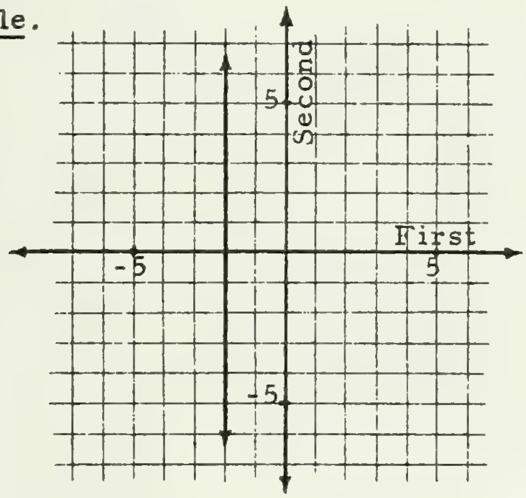
12. $\{(x, y): x \le 1 \text{ and } y \ge 2\}$

13. $\{(x, y): x \le 1 \text{ or } y \ge 2\}$

[Supplementary exercises are in Part F on page 4-122.]

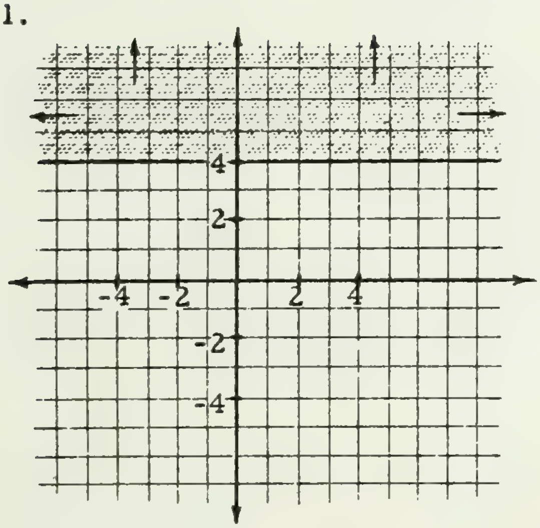
__B__. For each set pictured below, write its description.

__Sample__.

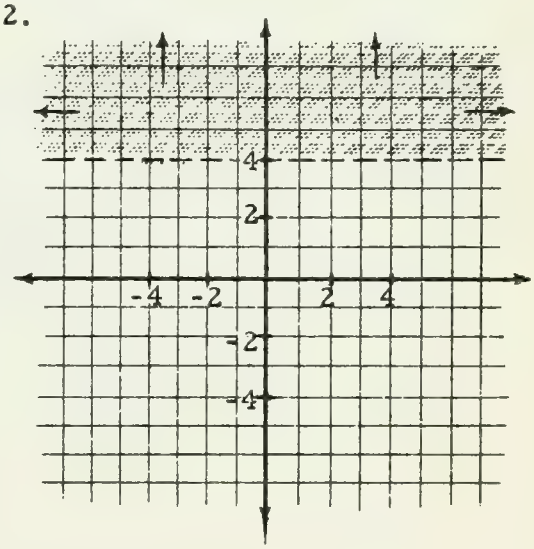


__Solution__. $\{(x, y): x = -2\}$

1.

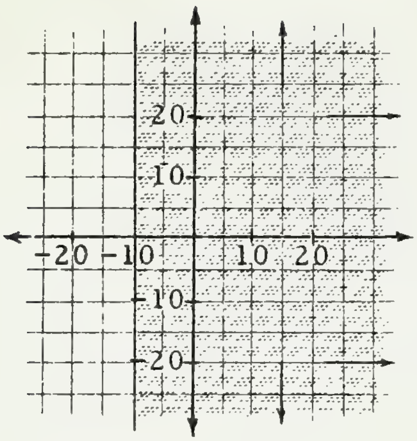


2.

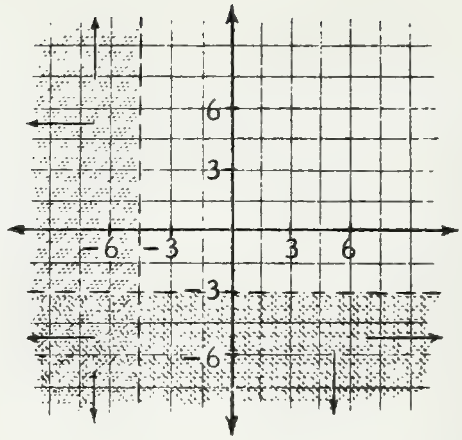


(continued on next page)

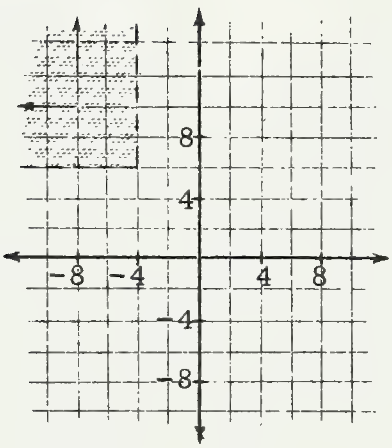
3.



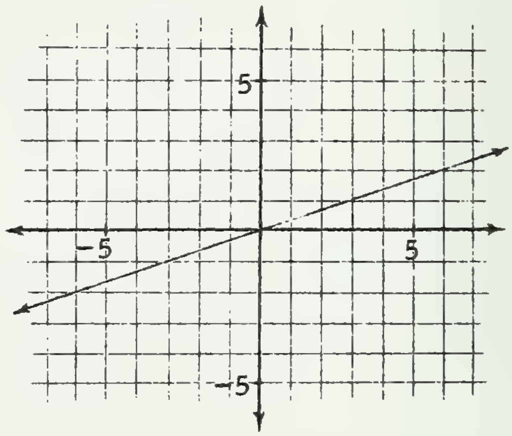
4.



5.



6.



[Supplementary exercises are in Part G, pages 4-122 and 4-123.]

LOCUS AND GRAPH

You will recall from Unit 3 that an open sentence such as:

$$x + 5 > 9$$

is satisfied by many numbers. The set of such numbers is the <u>solution set</u> or <u>locus</u> of '$x + 5 > 9$'. The graph [on a number line picture] of '$x + 5 > 9$' is the picture made up of the graphs of the numbers in the solution set.

An open sentence such as:

$$m + 2k = 9$$

can also be satisfied. For example, we can get a true sentence from it by substituting '7' for 'm' and '1' for 'k'. If we wish to report this fact,

it is not enough to say that the two numbers 1 and 7 satisfy the sentence [Why?]. Here, as in the Introduction, the notion of ordered pair is helpful. If we agree to consider 'm' as the "first" pronumeral and 'k' as the "second" pronumeral then what we mean will be clear if we say that the ordered pair (7, 1) satisfies the sentence. Under this agreement we say that the locus of the sentence:

$$m + 2k = 9$$

is

$$\{(m, k): \ m + 2k = 9\}.$$

Without such an agreement it would make no sense to ask what the solution set of '$m + 2k = 9$' is. However, we can indicate the agreement and, at the same time, ask the proper question by saying:

What is the solution set <u>in</u> <u>(m, k)</u> of '$m + 2k = 9$'?

One answer to this question is the name:

$$\{(m, k): \ m + 2k = 9\}.$$

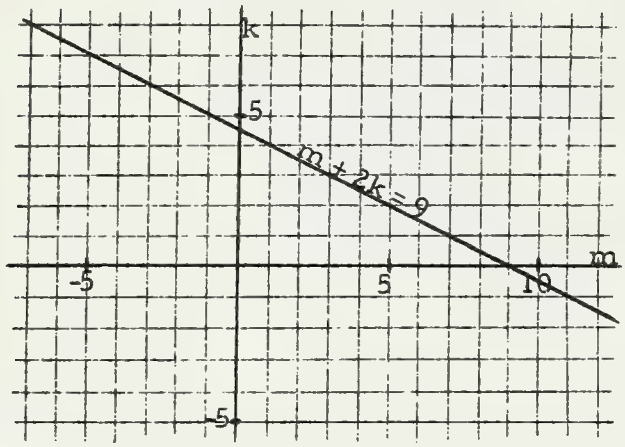
Another name for this locus is:

$$\{(x, y): \ x + 2y = 9\},$$

and still another name is:

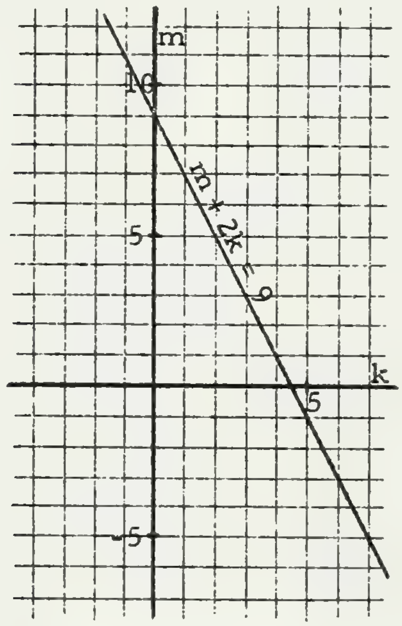
$$\{(k, m): \ k + 2m = 9\}.$$

The graph of a sentence is made up of graphs of the ordered pairs which belong to the solution set of the sentence. So, to draw the <u>graph</u> <u>in</u> <u>(m, k)</u> of '$m + 2k = 9$', you draw a picture of $\{(x, y): \ x + 2y = 9\}$. In drawing such a picture we follow our convention of using a horizontal line to picture the first component axis and a vertical line for the second. It is also customary to use the "first" pronumeral as a label at the right (or "positive") end of the picture of the first component axis, and the "second" pronumeral as a label at the upper (or "positive") end of the picture of the second component axis.



This is a picture (of part) of
$\{(x, y): x + 2y = 9\}$.

The diagram above shows the graph in (m, k) of 'm + 2k = 9'. Compare it with the following diagram which shows the **graph in** (k, m) of 'm + 2k = 9'.



This is a picture (of part) of
$\{(y, x): x + 2y = 9\}$.
[It is also a picture (of part) of
$\{(x, y): y + 2x = 9\}$.]

[4-31]

4/65

*

[For more exercises like those in Part B and for a discussion of some of the pedagogical problems involved in graphing equations and inequations, see "Graphing in Elementary Algebra" in the April 1956 issue of The Mathematics Teacher, and "Some Implications of Twentieth Century Mathematics for High Schools" in the February 1958 issue of The Mathematics Teacher.]

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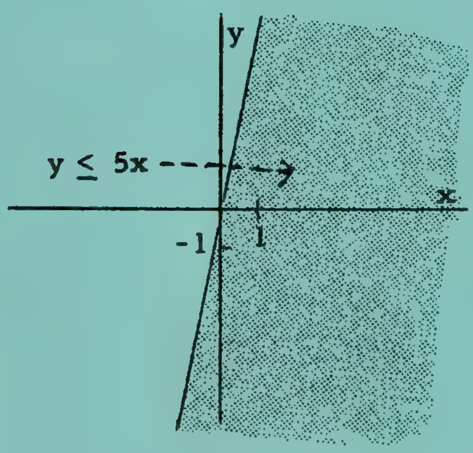
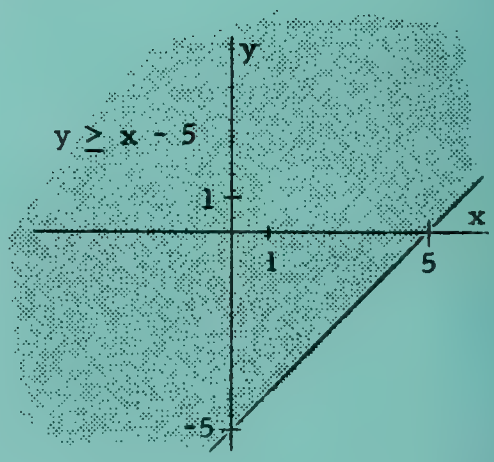
Answers for Part B [on pages 4-31 and 4-32].

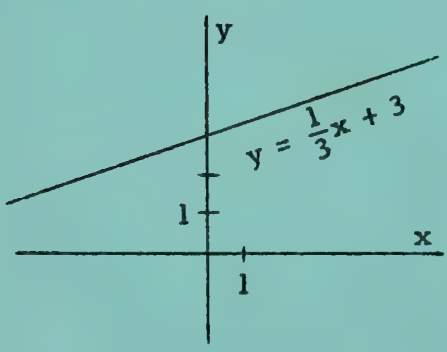
1.



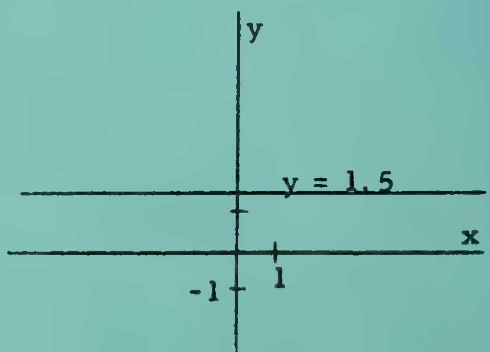
2.



3.



4.



[4-31]

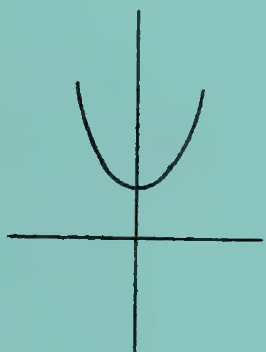
Next, consider equations which are satisfied by ordered pairs having the same second component. [Examples: $y = 2$ and: $y = 0x + 117.$] Now, students should be ready to consider these three types of equations, and their graphs:

(a) those satisfied by ordered pairs some two of which have the same first component;

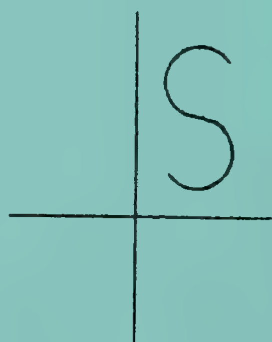
(b) those satisfied by ordered pairs some two of which have the same second component;

(c) those satisfied by ordered pairs no two of which have the same first component or the same second component.

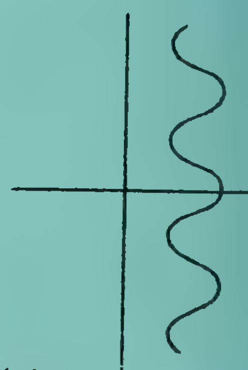
Ask the students to identify the type illustrated by these graphs:



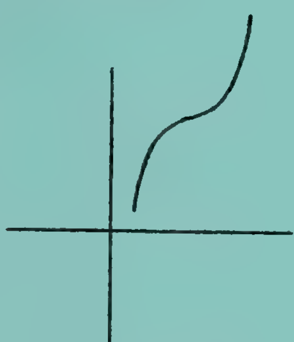
Answers: (b)



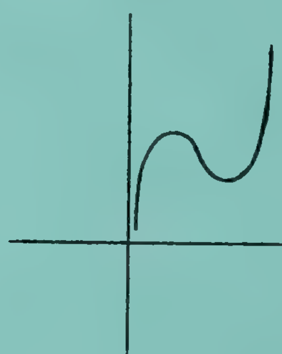
(a) and (b)



(a)



Answers: (c)



(b)

(c)

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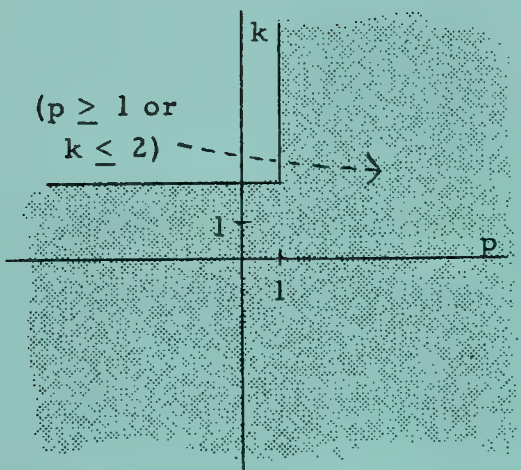
7.



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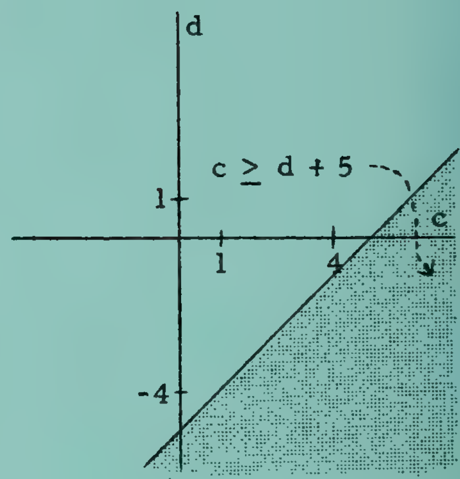

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You may want to introduce the table-of-values device in connection with Part B. If you do this, we recommend a vertical arrangement with first components listed in the left-hand column.

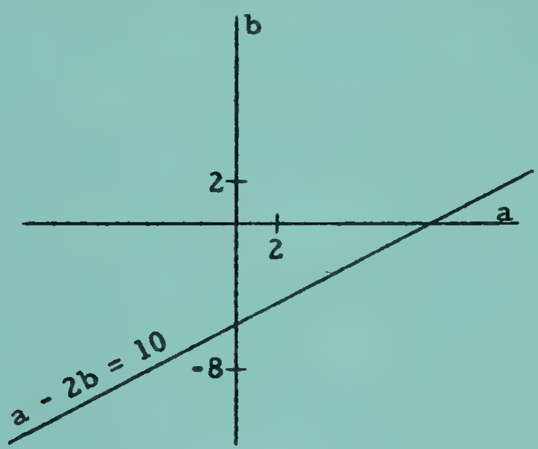
*

As you discuss the exercises in Part B, ask questions which will emphasize the idea that the graph of a sentence covers those dots and only those dots whose coordinates satisfy the given sentence. This is because the solution set of a sentence contains those points and only those points whose components satisfy the sentence. Consider the equation '$y = (1/3)x + 12$'. Ask students whether (300, 112) is an element of the solution set. If the reply is 'yes', then the graph of (300, 112) is on the graph of the given equation. Next, consider (270, 102); (1530, 522); (612, 214). When someone explains that (612, 214) is not an element of the solution set, ask where the graph of (612, 214) is with respect to the graph of the given equation. Then, ask about (300, 9720). A bonus to the student who explains that since they have already agreed that (300, 112) <u>is</u> an element of the solution set, then an ordered pair with the same first component and a different second component could not possibly be an element of the solution set for <u>this</u> equation!

Ask the students to give some equations which could be satisfied by ordered pairs all of which have the same first component. [Examples are: $x = 0y$, and: $x = 3$.]

Answers for Part A. [We assume that all figures are infinite in extent.]

1.



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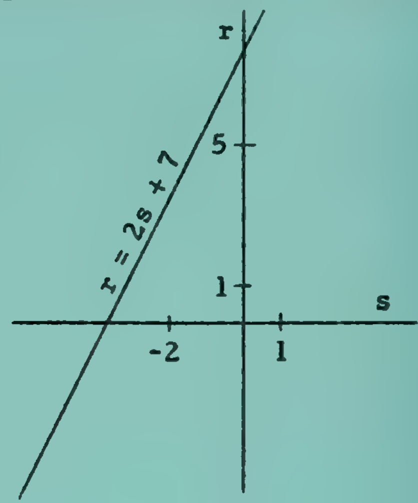

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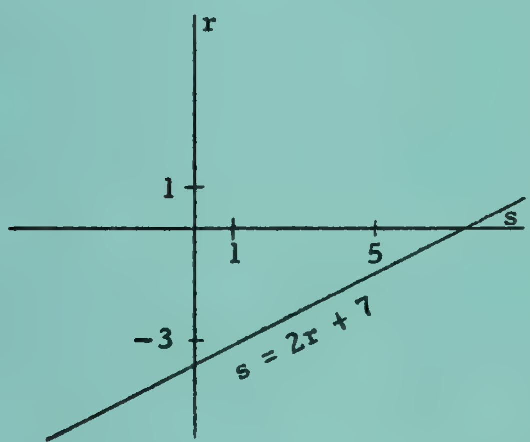

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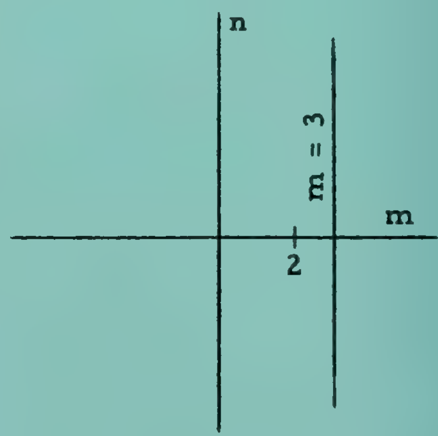

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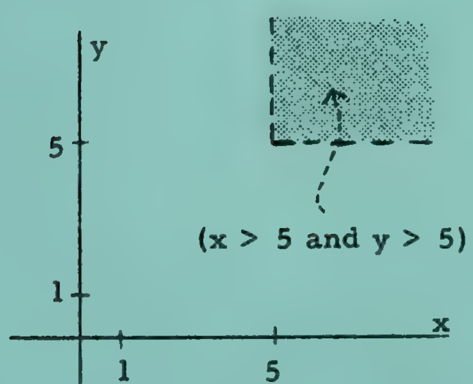

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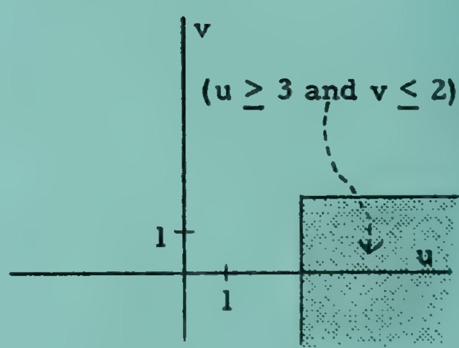

EXERCISES

A. Make labeled pictures of these loci.

1. the locus in (a, b) of '$a - 2b = 10$'

2. the locus in (s, r) of '$r = 2s + 7$'

3. the locus in (s, r) of '$s = 2r + 7$'

4. the locus in (m, n) of '$m = 3$'

5. the locus in (x, y) of '$x > 5$ and $y > 5$'

6. the locus in (u, v) of '$u \geq 3$ and $v \leq 2$'

7. the locus in (p, k) of '$p \geq 1$ or $k \leq 2$'

8. the locus in (c, d) of '$c \geq d + 5$'

* * *

In dealing with sentences which contain the pronumerals 'x' or 'y', it is customary to regard 'x' as the "first" pronumeral and 'y' as the "second" pronumeral unless the contrary is stated. So, for example, when one says:

the locus of '$3x + 2y = 5$'

he means

the locus in (x, y) of '$3x + 2y = 5$'.

This convention is widely observed, and, indeed, the first component axis is often called <u>the x-axis</u>, the second component axis, then, being called <u>the y-axis</u>. In addition, we say that the abscissa of a dot is its <u>x-coordinate</u>, and the ordinate of a dot is its <u>y-coordinate</u>. The number plane is also called <u>the (x, y)-plane</u>.

* * *

B. Graph each of the following sentences. [That is, draw a picture of its number plane locus.]

1. $y \leq 5x$

2. $y \geq x - 5$

3. $y = \frac{1}{3}x + 3$

4. $y = 1.5$

(continued on next page)

5. $x \geq -3$

6. $x + y \leq 2$

7. $3x = y$

8. $y = x + 1$

9. $x + y = 0$

10. $2x + 3 = y$

11. $y = x - 3$

12. $2x - 8 = 0$

13. $y + 5 = 0$

14. $10x + 4y = 0$

15. $y = 5x$

16. $3y = 4x + 1$

17. $3y + 2x = 6$

18. $3y - 2x = 6$

19. $x^2 = 9$

20. $x^2 \leq 9$

21. $|y| < 3$

22. $|xy| > 0$

23. $x + y = 7 + x$

24. $xy = 0$

25. $x = 3$ and $y = 2$

26. $x = 3$ or $y = 2$

27. $x = 3$ and $y > 2$

28. $x = 3$ or $y > 2$

29. $y = 7$ or $y = 5$ or $y = 3$

30. $y = 7$ and $y = 5$ and $y = 3$

31. $|x| + |y| = 10$

32. $|x| + |y| \leq 10$

33. $x + y = y + x$

34. $x - y = y - x$

35. $x^2 + y^2 = 0$

36. $x^2 - y^2 = 0$

37. $xy = yx$

38. $x \div y = y \div x$

39. $\frac{x}{x} + \frac{y}{y} = 2$

C. Use the pages at the end of the unit, and plot the points in each of the four regions which belong to the locus of each of the following sentences.

1. $x = 1\ 000\ 000$

2. $y = -450\ 002$

3. $x > y$

4. $y < 1$

5. $x = y$

6. $y \geq x$

7. $y = -13$

8. $x < y$

9. $y = 2x$

10. $x + y = y + x$

into

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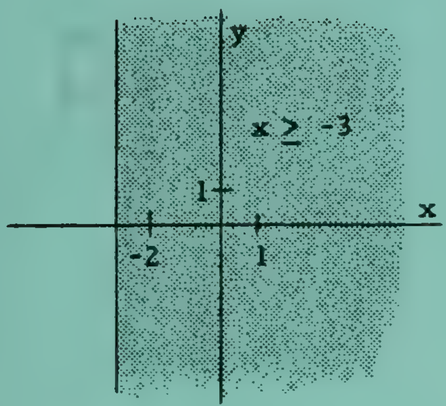
5.



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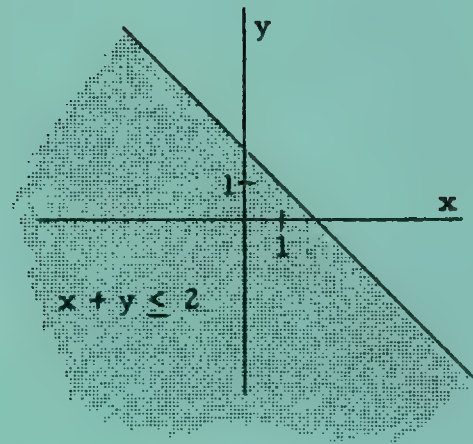

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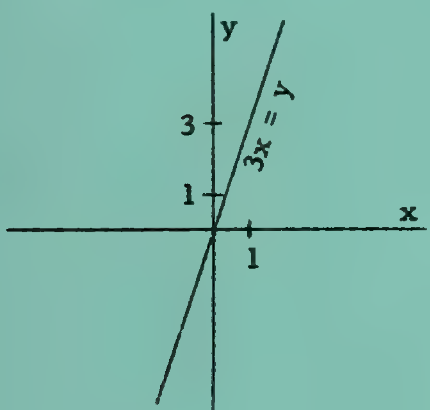

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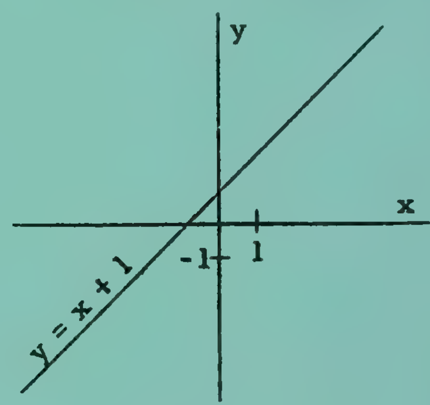

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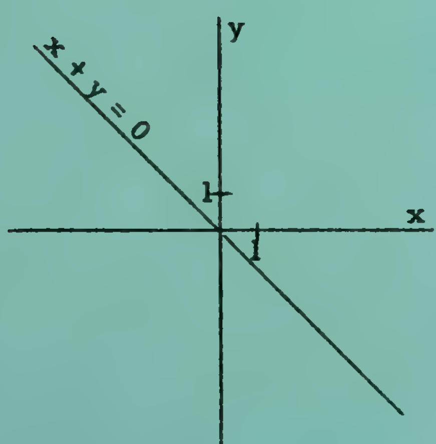

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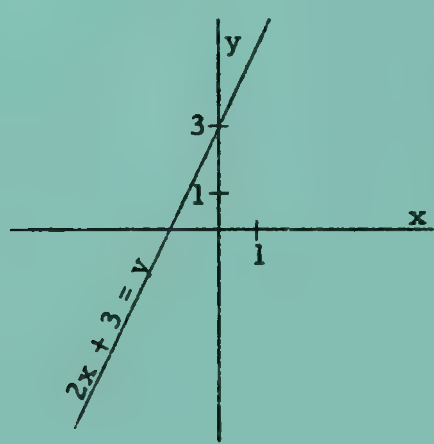

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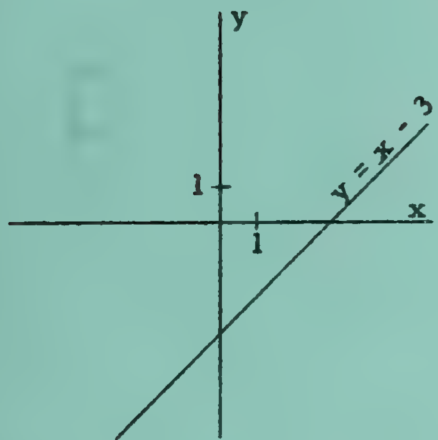
11.



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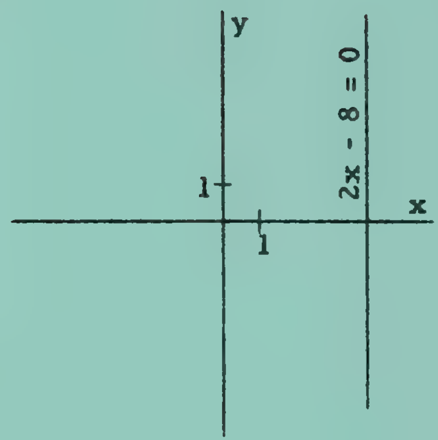

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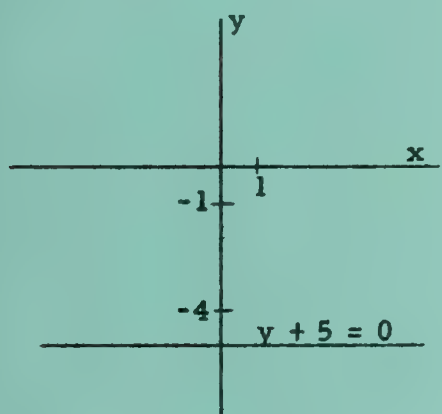

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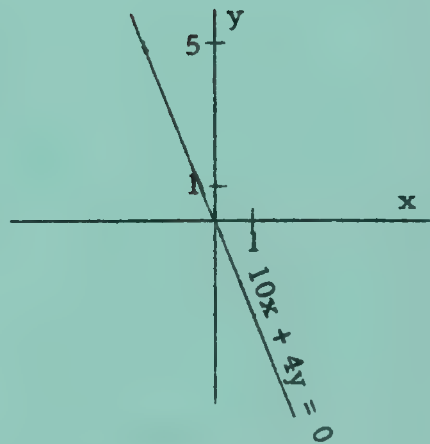

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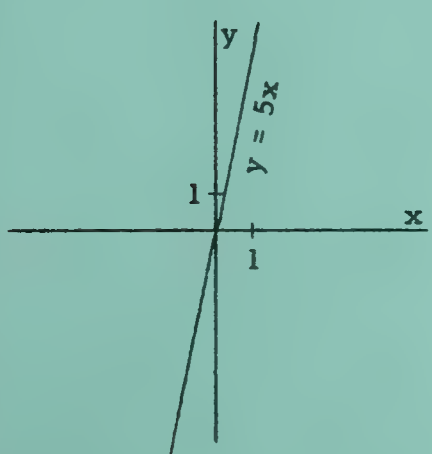

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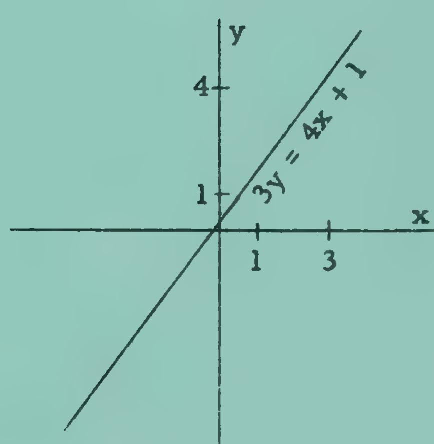

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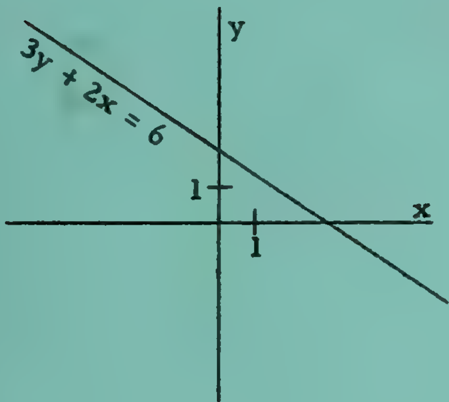
17.



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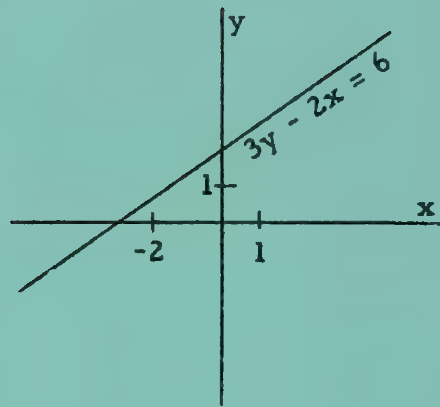

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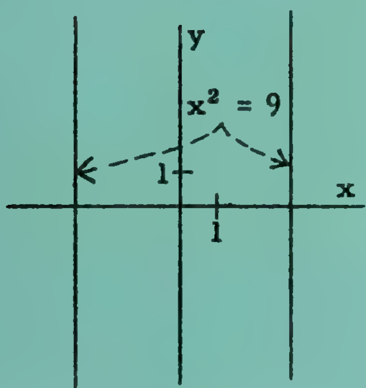

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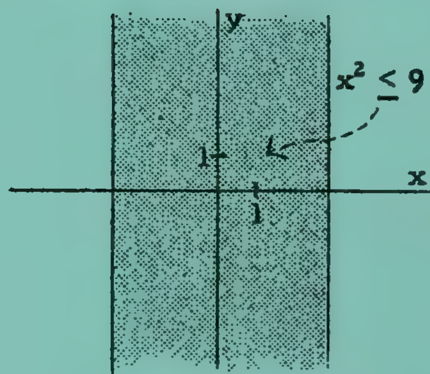

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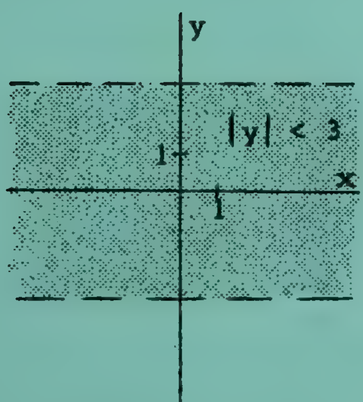

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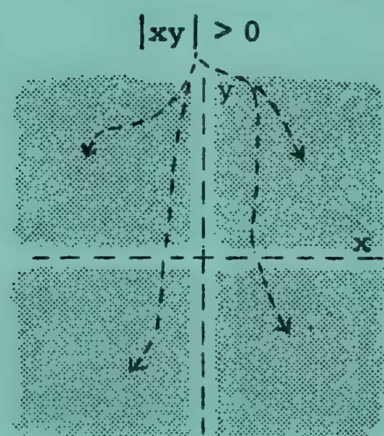

[The number plane, excluding the axes.]

[4-32]

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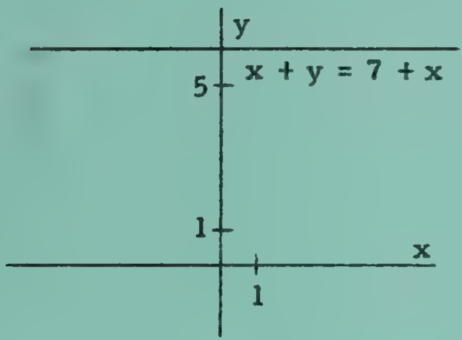
23.



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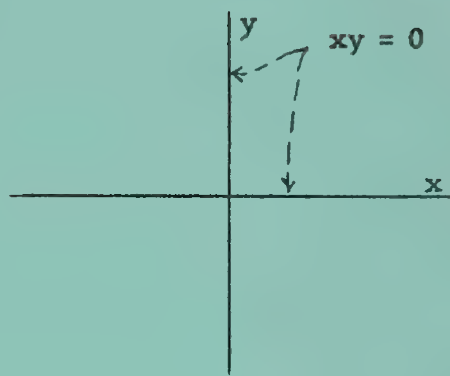
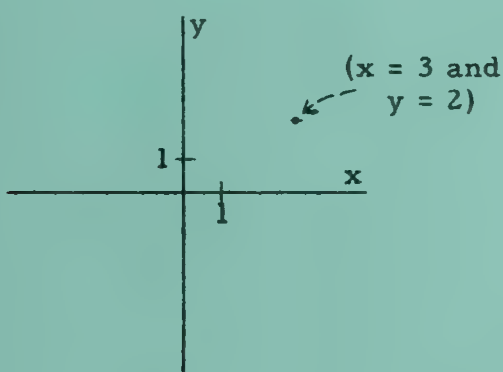


[In Exercises 25 through 30 note again the importance of distinguishing between the use of 'and' and the use of 'or'. For example, the graph in Exercise 29 consists of 3 parallel lines, and the graph in Exercise 30 is the empty set.]

25.



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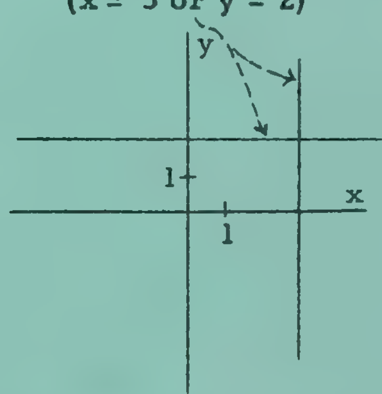

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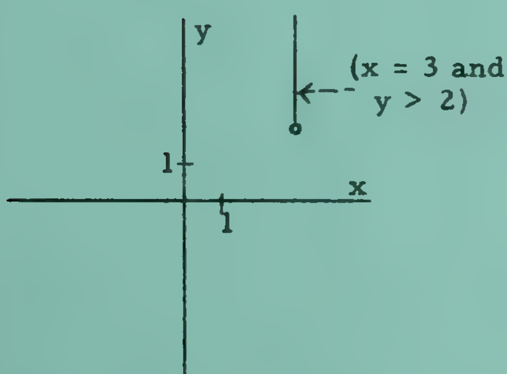

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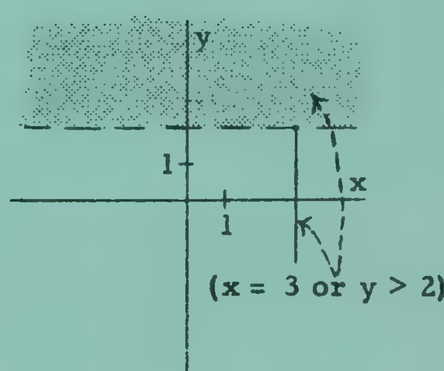

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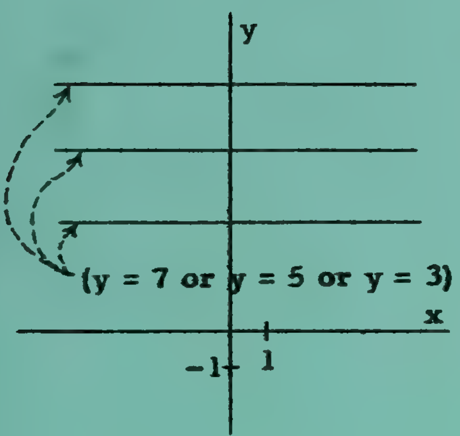
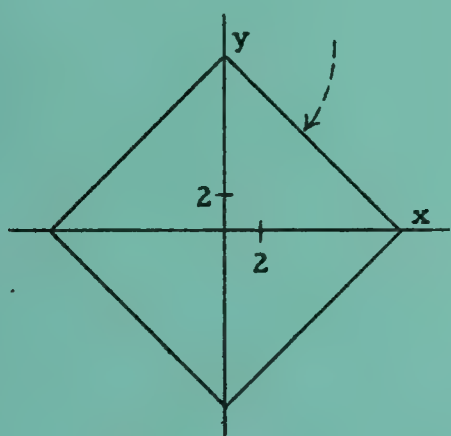
29.

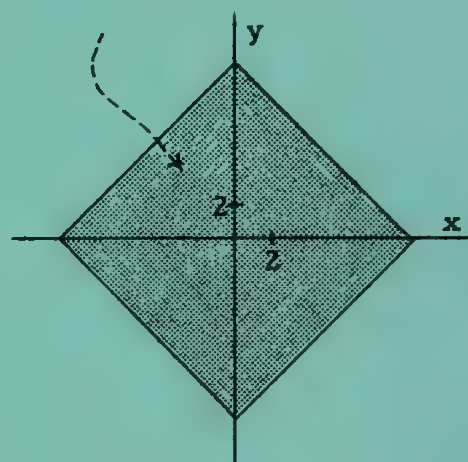


30. The empty set, since there is no point whose second component is 7, 5, and 3.

31. $|x| + |y| = 10$

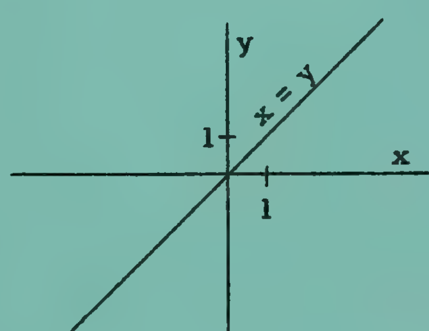


32. $|x| + |y| \leq 10$



33. The graph will picture the number plane itself.

34.



35. The graph will picture just the origin. [0 is the only number whose square is not positive.]

36. The graphs of 'x = y' [See Ex. 34], and of 'x = −y' [See Ex. 9], on the same picture.

[4-32]

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37. The graph will picture the number plane itself.
38. The graph will be the same as that for Exercise 36 except that the origin will be excluded.
39. The graph will picture the number plane with the component axes excluded.

*

Answers for Part C.

[We describe the graph for each exercise, rather than giving the picture, in order to conserve space.]

1. A vertical line in B, and a vertical line in C.
2. A horizontal line in C.
3. A shaded region below the 'x = y'-line in B; also, all of C.
4. A shaded region covering the part of A below the 'y = 1'-line; also, all of C.
5. A diagonal line in B.
6. All of A; a shaded region above the 'x = y'-line in B; all of D.
7. A horizontal line in A [the last line in the picture].
8. All of A; a shaded region above the 'x = y'-line in B; all of D.
9. No part of the graph of 'y = 2x' can be shown on any of the four pictures.
10. The graph of 'x + y = y + x' will cover all four pictures.

*

Review Quiz.

1. Draw a picture of the lattice

   $$\{-3, -2, -1, 0, 1, 2, 3\} \times \{-3, -2, -1, 0, 1, 2, 3\}.$$

   How many points are there in this lattice with

   (a) first component -2? (b) second component $+3$?

   (c) first component ≥ 3? (d) second component ≤ -3?

   (e) first component > 2 and second component < 3?

   (f) first component = second component?

[4-32]

2. List the points of the lattice in Exercise 1 which belong to

(a) $\{(x, y): x = y + 2\}$ (b) $\{(x, y): x = 2y\}$

(c) $\{(x, y): y = 3x - 2\}$ (d) $\{(x, y): x = 2y - 1\}$

3. When you are throwing a green die and a white die, and agree that the green die gives the first number of an ordered pair while the white die gives the second number of an ordered pair, what are the chances that when you make a throw you will get

(a) (3, 2)? (b) (5, 6)? (c) (2, 5) or (1, 4)?

4. The expression '$(a - 7)(a + 9)$' is equivalent to which of the following?

(a) $a^2 - 63$ (b) $a^2 + 2$ (c) $a^2 + 2a - 63$

(d) $a^2 - 2a - 63$ (e) $a^2 - 16a - 63$

5. In a certain number plane lattice game, moves are made according to the following rule:

$$(p, q) \to (-p, -3q).$$

Start at the graph of $(-3, 5)$. At the end of three moves you should be at the graph of which of these ordered pairs?

(a) $(-3, 135)$ (b) $(3, 135)$ (c) $(-3, 15)$

(d) $(3, -45)$ (e) $(3, -135)$

6. Which of the following equations has roots 1 and -1?

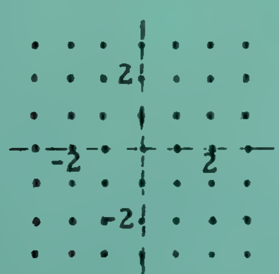
(a) $7.5b^2 + 7.5 = 0$ (b) $3b^2 - 3 = 0$ (c) $\frac{1}{5}a^2 = \frac{1}{25}$

(d) $.5x^2 = 0$ (e) $\frac{1}{4}n^2 = 4$

*

Answers for Quiz.

1.



(a) 7 (b) 7 (c) 7 (d) 7 (e) 6 (f) 7

2. (a) $(-1, -3)$, $(0, -2)$, $(1, -1)$, $(2, 0)$, $(3, 1)$

(b) $(-2, -1)$, $(0, 0)$, $(2, 1)$ (c) $(0, -2)$, $(1, 1)$

(d) $(3, 2)$, $(1, 1)$, $(-1, 0)$, $(-3, -1)$

3. (a) 1/36 (b) 1/36 (c) 1/18

4. [(c)] 5. [(e)] 6. [(b)]

[4-32]

into

The union of the four quadrants is the set consisting of those points which belong to neither component axis, that is, $\{(x, y): x \neq 0$ and $y \neq 0\}$. [What is $\{(x, y): x \neq 0$ or $y \neq 0\}$?] The union of the four quadrants, the x-axis, and the y-axis is the number plane, that is, $\{(x, y): x = x\}$. The intersection of any two quadrants is $\emptyset$. So is the intersection of any quadrant and an axis. The intersection of the component axes is $\{(0, 0)\}$, that is, the set whose only member is the origin.

*

The exercises in Part D are designed to give the students practice in predicting the orientation of the graph of an equation. For a solution set to contain points in Quadrant I, it must contain ordered pairs with both components positive. So, for example, in Exercise 1(a) the student asks himself if there are positive values of 'x' and 'y' which satisfy '$y = -2x + 3$'. [See the 12 exercises at the bottom of TC[4-35]e.]

*

Answers for Part D [on pages 4-33 and 4-34].

1. (a) yes, yes, no, yes (b) yes, yes, no, yes
   (c) yes, yes, no, no (d) yes, yes, yes, no
   (e) yes, yes, yes, no
2. (a) no, yes, yes, yes, (b) no, yes, yes, yes
   (c) no, no, yes, yes, (d) yes, no, yes, yes.
   (e) yes, no, yes, yes
3. (a) no, no, no, no (b) no, no, no, no
4. (a) yes, no, no, yes (b) yes, no, yes, yes
   (c) yes, yes, no, yes
5. (a) no, yes, yes, no (b) yes, yes, yes, no
   (c) no, yes, yes, yes,
6. yes, yes, yes, yes
7. yes, yes, yes, yes

*

One of our pilot school teachers suggested the following two exercises for inclusion in Part D.

8. $y = 3x$ 9. $y = -5x$

Answers for Exercise 8: yes, no, yes, no

Answers for Exercise 9: no, yes, no, yes

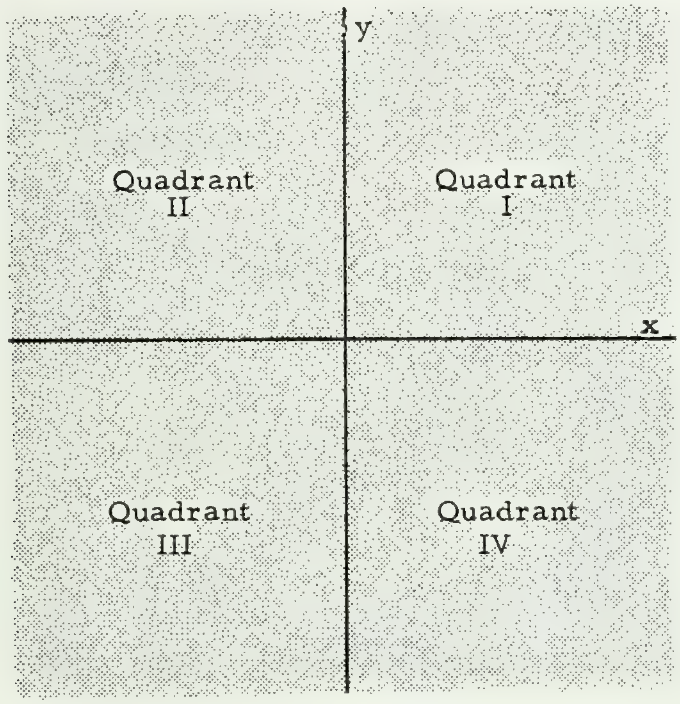
<u>D</u>. The x- and y-axes separate the points in the (x, y)-plane into 4 regions called <u>quadrants</u>. Each quadrant is a set of points.

Quadrant I is $\{(x, y): x > 0 \text{ and } y > 0\}$.

Quadrant II is $\{(x, y): x < 0 \text{ and } y > 0\}$.

Quadrant III is $\{(x, y): x < 0 \text{ and } y < 0\}$.

Quadrant IV is $\{(x, y): x > 0 \text{ and } y < 0\}$.



What is the union of the four quadrants? What is the union of the four quadrants and the x- and y-axes? What is the intersection of any two quadrants? What is the intersection of any quadrant and an axis? What is the intersection of the x-axis and the y-axis?

(continued on next page)

Look at the following table. For each of the sentences in the left-hand column, tell which quadrants contain points of its solution set. Try to answer these questions without making drawings. [The first exercise has been completed for you. Check it.]

	Quadrant			
	I	II	III	IV
1. (a) $y = -2x + 3$	yes	yes	no	yes
(b) $y = -x + 3$				
(c) $y = 3$				
(d) $y = x + 3$				
(e) $y = 2x + 3$				
2. (a) $y = -2x - 3$				
(b) $y = -x - 3$				
(c) $y = -3$				
(d) $y = x - 3$				
(e) $y = 2x - 3$				
3. (a) $y = 0$				
(b) $x = 0$				
4. (a) $x = 4$				
(b) $x = y + 4$				
(c) $x = -y + 4$				
5. (a) $x = -4$				
(b) $x = y - 4$				
(c) $x = -y - 4$				
6. $x^2 + y^2 = 25$				
7. $\lvert x \rvert = 1$ and $\lvert y \rvert = 1$				

[4-35]

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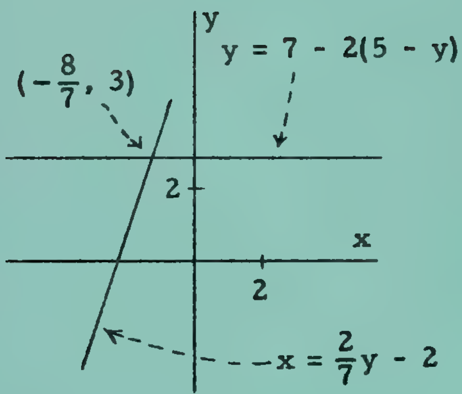
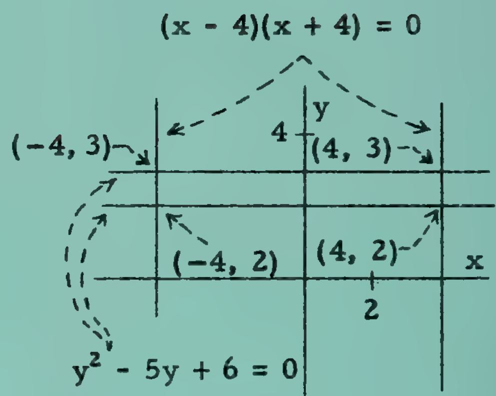
15.



16.



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15. $\{(x, y): y = 7 - 2(5 - y)\} \cap \{(x, y): x = \frac{2}{7}y - 2\} = \{(-\frac{8}{7}, 3)\}$

16. $\{(x, y): (x - 4)(x + 4) = 0\} \cap \{(x, y): y^2 - 5y + 6 = 0\} =$
$\{(4, 3), (-4, 3), (-4, 2), (4, 2)\}$

*

One of our pilot school teachers suggested that the following problem be included.

(a) $|x - 3| = y$ (b) $|2x| = y$

*

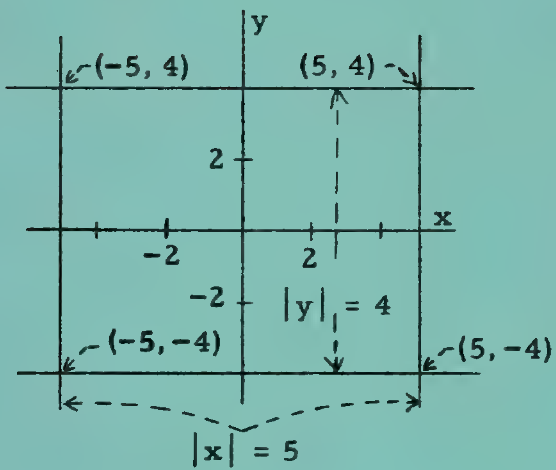
Here are supplementary exercises to use. After the students have drawn the graphs of these equations, use questions to determine whether they have discovered the idea of "slope" and "y-intercept" which can be determined by examination of the equation.

Draw the graphs of each pair of equations on the same number plane picture. What do you notice about the pair of graphs?

1. $y = -3x + 2$ $y = 3x + 2$	2. $y = (1/2)x - 5$ $y = -(1/2)x - 5$	3. $y = x + 6$ $y = -x + 6$
4. $y = 2x - 4$ $y = 2x + 4$	5. $y = -(3/2)x + 3$ $y = -(3/2)x - 3$	6. $y = x - 10$ $y = -x + 10$
7. $x = y + 7$ $x = -y + 7$	8. $x = -2y + 5$ $x = 2y + 5$	9. $y = (1/3)x - 6$ $y = -(1/3)x - 6$
10. $x = 3y + 4$ $x = 3y - 4$	11. $x = y + 10$ $x = y - 10$	12. $x = (5/4)y - 8$ $x = (5/4)y + 8$

[4-35]

11.



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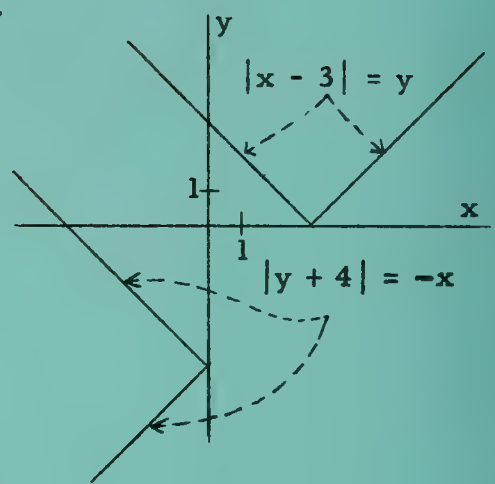

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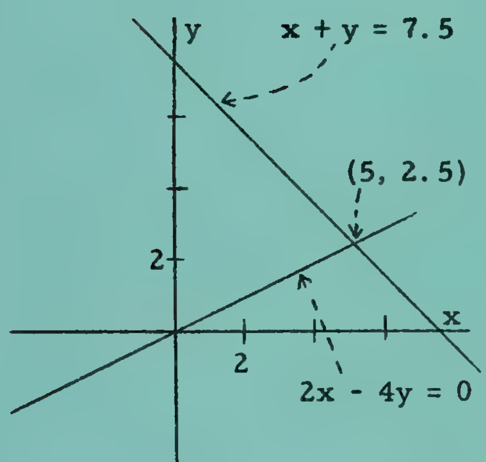

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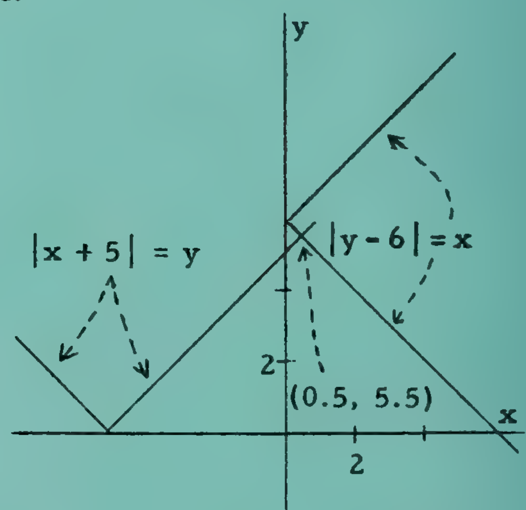

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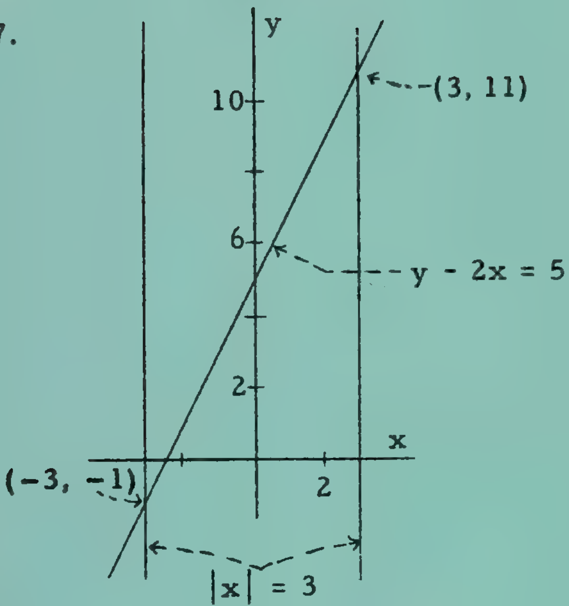
11. $\{(x, y): |y| = 4\} \cap \{(x, y): |x| = 5\} = \{(5, 4), (-5, 4), (-5, -4), (5, -4)\}$
12. $\{(x, y): |x - 3| = y\} \cap \{(x, y): |y + 4| = -x\} = \emptyset$, [that is, there are no points in the intersection of the solution sets].
13. $\{(x, y): x + y = 7.5\} \cap \{(x, y): 2x - 4y = 0\} = \{(5, 2.5)\}$
14. $\{(x, y): |x + 5| = y\} \cap \{(x, y): |y - 6| = x\} = \{(0.5, 5.5)\}$

*

7.



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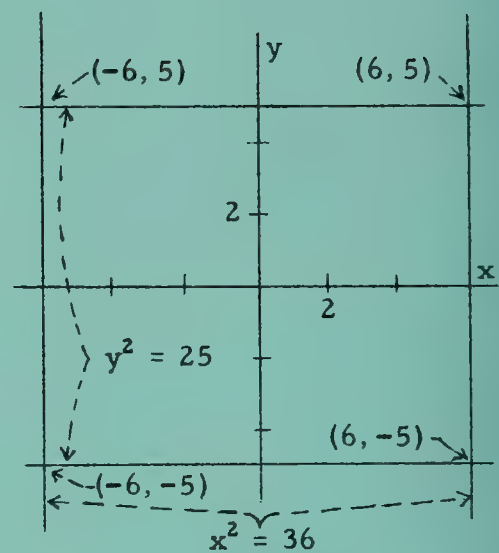

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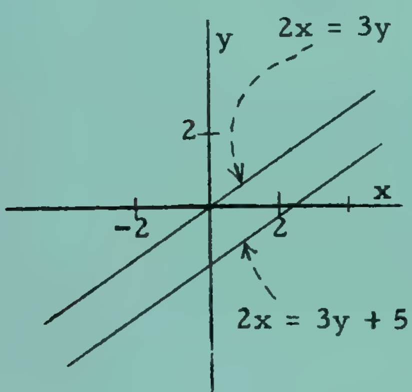

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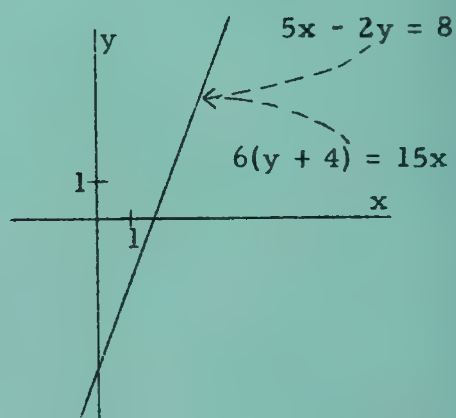

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7. $\{(x, y): |x| = 3\} \cap \{(x, y): y - 2x = 5\} = \{(3, 11), (-3, -1)\}$
8. $\{(x, y): y^2 = 25\} \cap \{(x, y): x^2 = 36\} = \{(6, 5), (-6, 5), (-6, -5), (6, -5)\}$
9. $\{(x, y): 2x = 3y\} \cap \{(x, y): 2x = 3y + 5\} = \emptyset$, [that is, there are no points in the intersection of the solution sets].
10. Since '$5x - 2y = 8$' and '$6(y + 4) = 15x$' are equivalent equations, they have the same solution set. Hence, each ordered pair which is a member of $\{(x, y): 5x - 2y = 8\}$ is also a member of $\{(x, y): 6(y + 4) = 15x\}$, and conversely.

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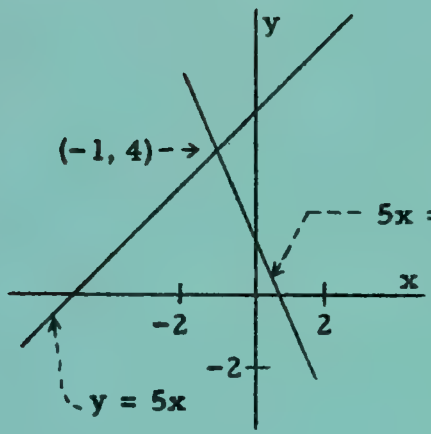
[4-35]

3.



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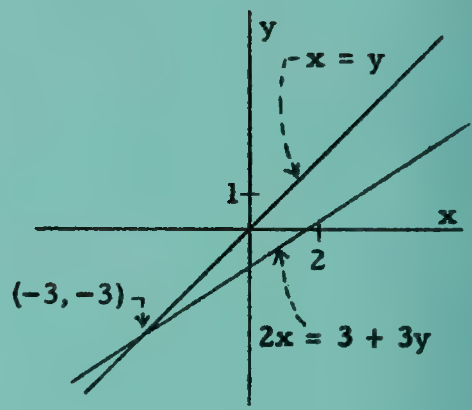

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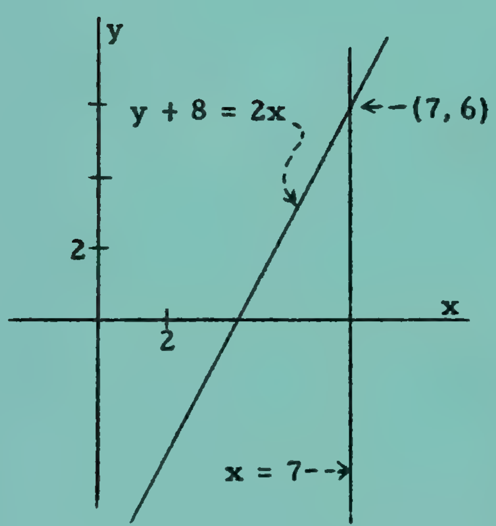

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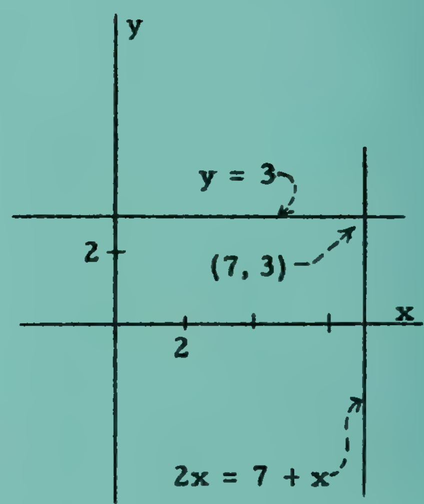

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3. $\{(x, y): 5x = 3 - 2y\} \cap \{(x, y): y = 5 + x\} = \{(-1, 4)\}$
4. $\{(x, y): x = y\} \cap \{(x, y): 2x = 3 + 3y\} = \{(-3, -3)\}$
5. $\{(x, y): x = 7\} \cap \{(x, y): y + 8 = 2x\} = \{(7, 6)\}$
6. $\{(x, y): y = 3\} \cap \{(x, y): 2x = 7 + x\} = \{(7, 3)\}$

*

[4-35]

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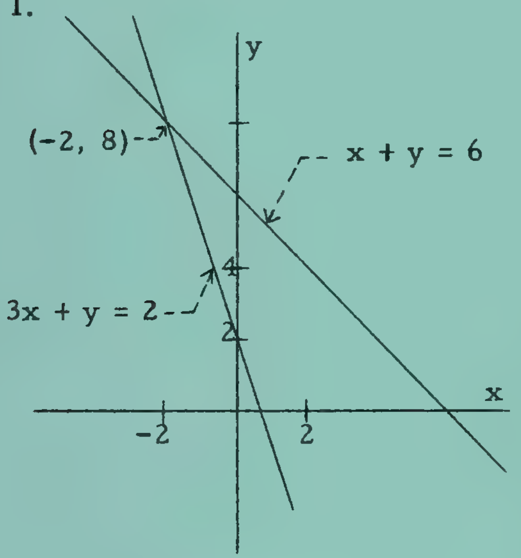
For Part E, some students will come up with algebraic techniques for finding points of intersection. This should not be discouraged. However, there is a complete treatment in Unit 5 of the topic of systems of two linear equations in two pronumerals.

[Beginning with this group of graphs, we are typing the equation in a horizontal position, rather than along its graph. [We do this to save typing time!] When your students draw the graphs, it is preferable for the equation to be written along its graph.]

Answers for Part E.

1.



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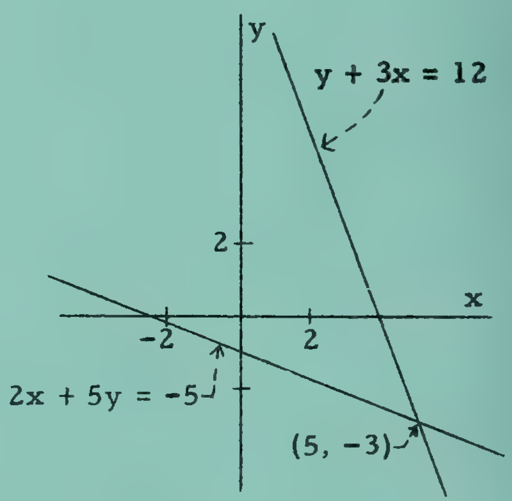


The answers we expect for these exercises are pictures consisting of labeled graphs and labeled points of intersection. In order to help you in correcting abuses of language such as '$x + y = 6$ and $3x + y = 2$ intersect in $(-2, 8)$' or '$x + y = 6$ and $3x + y = 2$ intersect in $x = -2$ and $y = 8$', we give here a correct way of expressing what is meant. Do not insist on students writing these statements in addition to making the pictures.

1. $\{(x, y): x + y = 6\} \cap \{(x, y): 3x + y = 2\} = \{(-2, 8)\}$

2. $\{(x, y): y + 3x = 12\} \cap \{(x, y): 2x + 5y = -5\} = \{(5, -3)\}$

E. Each of the following exercises contains a pair of equations. Graph each of the two equations and give the ordered pairs which are in the intersection of their solution sets.

1. (a) $x + y = 6$
   (b) $3x + y = 2$

2. (a) $2x + 5y = -5$
   (b) $y + 3x = 12$

3. (a) $5x = 3 - 2y$
   (b) $y = 5 + x$

4. (a) $x = y$
   (b) $2x = 3 + 3y$

5. (a) $x = 7$
   (b) $y + 8 = 2x$

6. (a) $y = 3$
   (b) $2x = 7 + x$

7. (a) $|x| = 3$
   (b) $y - 2x = 5$

8. (a) $yy = 25$
   (b) $xx = 36$

9. (a) $2x = 3y$
   (b) $2x = 3y + 5$

10. (a) $5x - 2y = 8$
    (b) $6(y + 4) = 15x$

11. (a) $|x| = 5$
    (b) $|y| = 4$

12. (a) $|x - 3| = y$
    (b) $|y + 4| = -x$

13. (a) $x + y = 7.5$
    (b) $2x - 4y = 0$

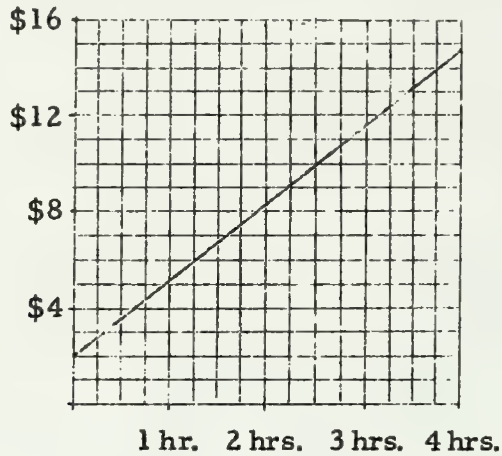
14. (a) $|x + 5| = y$
    (b) $|y - 6| = x$

15. (a) $y = 7 - 2(5 - y)$
    (b) $x = \frac{2}{7}y - 2$

16. (a) $(x - 4)(x + 4) = 0$
    (b) $y^2 - 5y + 6 = 0$

[Supplementary exercises are in Part H, pages 4-123 and 4-124.]

4.03 Graphs of formulas.--A repairman uses the following procedure in charging for service calls. He charges $2.00 for going to a home, and he charges $3.20 more for each hour that he works. He could use the following chart to determine the amount to charge.



Use the chart to answer these questions.

(1) What is the charge if the repairman works 2 hours?

(2) What is the charge if the repairman works $3\frac{1}{2}$ hours?

(3) What is the charge if the repairman works 45 minutes?

(4) How long did the repairman work if the charge was $6.25?

(5) How long did the repairman work if the charge was $2.00?

(6) How long did the repairman work if the charge was $1.00?

The repairman could also use a formula instead of a chart to compute his charge. If he uses a 'c' to hold a place for a numeral which tells his charge and a 't' to hold a place for a numeral for the number of hours he works, then he can use the equation:

$$c = 2.00 + 3.20t$$

as a formula for computing his charges. [The equation serves as a formula when one has decided on the "use" of the pronumerals 'c' and 't'.] For example, if he worked 2 hours, he would replace 't' by '2' and obtain:

$$\begin{aligned} c &= 2.00 + 3.20(2) \\ &= 2.00 + 6.40 \\ &= 8.40 \end{aligned}$$

which tells him that his charge for working 2 hours is $8.40.

In pages 4-36 through 4-42, the student is shown some of the uses for graphs.

*

Some students may object that the repairman would not keep track of his time to the nearest 15 minutes. This may be the case, but it should not concern us here. The idea is to learn to use a straight line graph in this type of problem. There may be some disagreement in (5) over how long the repairman worked if his charge was \$2.00. The idea we had in mind was that he was able to make the repair almost immediately and that he did not charge for any working time but merely for making the call. An impossible problem is given in (6). We give more attention to the question of meaningful problems later in this section.

*

Answers for questions.

(1) About \$8.50 (2) About \$13.00

(3) About \$4.50 (4) About 1 hour and 20 minutes

(5) Just a few minutes; perhaps he just gave advice and didn't actually work at all!

(6) Impossible. [He would not even have made a call!]

(7) \$14.80 (8) \$10.00 (9) \$6.00

(10) $\frac{1}{4}$ hour (11) Impossible.

*

On page 4-37, as in the discussion of worded problems in Unit 3, although the domain of the pronumerals 'c' and 't' is actually the set of numbers of arithmetic, we pretend that the domain is the set of real numbers and pay attention only to nonnegative results. So, in graphing '$c = 2.00 + 3.20t$', we would restrict our activities to $\{(x, y): x \geq 0 \text{ and } y \geq 0\}$.

4.03

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Use the formula:

$$c = 2.00 + 3.20t$$

to answer these questions.

(7) What is the charge if the repairman works 4 hours?

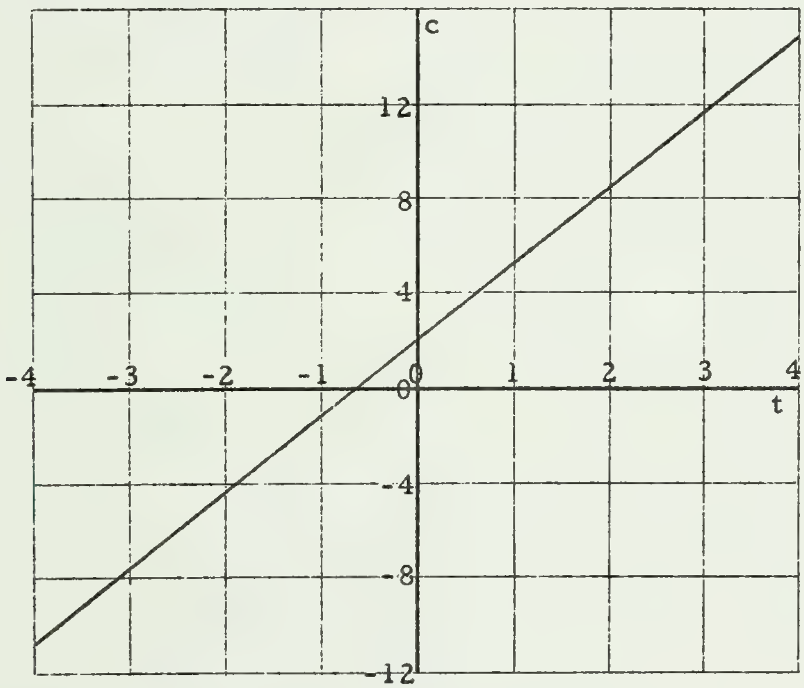
(8) What is the charge if the repairman works $2\frac{1}{2}$ hours?

(9) What is the charge if the repairman works 75 minutes?

(10) How long did the repairman work if the charge was \$2.80?

(11) How long did the repairman work if the charge was \$.40?

When you used the chart before to find charges, you were using the locus in (t, c) of '$c = 2.00 + 3.20t$'. Now, if you were told to graph '$c = 2.00 + 3.20t$' [using a c-axis for the vertical axis and a t-axis for the horizontal axis], you would get something like this.



This graph contains dots corresponding with some ordered pairs which satisfy the equation but which do not make sense in the problem to which the equation applies. For example, points in the second quadrant would correspond to negative numbers of hours worked. Points in the third quadrant would correspond to negative numbers of dollars charged and negative numbers of hours worked. Points in Quadrants II, III, and IV do not apply in this situation. Therefore, we should not even draw these quadrants [or the negative halves of the axes] when graphing '$c = 2.00 + 3.20t$' as a formula for finding repair charges.

Whenever you make a graph of a formula, you should keep in mind the kinds of numbers which enter into the problems to which the formula applies. In the repairman formula we are interested in charges and in time worked, both of which are measured by numbers of arithmetic. So, the domain of the pronumerals 'c' and 't' in 'c = 2.00 + 3.20t' is the set of numbers of arithmetic. [The domain of a pronumeral is the set of its values.] Now, in transforming an equation which contains pronumerals whose domain is the set of numbers of arithmetic, we usually find it convenient to pretend that their domain is the set of nonnegative real numbers. Then, in solving, we pay attention only to nonnegative results. Because the nonnegative real numbers behave like the numbers of arithmetic in computation, this reinterpretation of an equation will not lead us into error. Similarly, when graphing a formula whose pronumerals have the set of numbers of arithmetic as domain, we can think of the locus of the formula as a subset of the cartesian square of the set of nonnegative real numbers [$\overrightarrow{0, 1} \times \overrightarrow{0, 1}$]. So, for example, we speak of points in the first quadrant belonging to the locus of the formula even though points in the first quadrant have real number components.

EXERCISES

1. A formula for finding an approximation to the circumference of a circle is:

$$C = 6.28r.$$

Because the formula deals with measures of geometric figures, the domain of 'C' and of 'r' is the set of numbers of arithmetic. So, a graph of the formula is conveniently drawn on a picture of $\overrightarrow{0, 1} \times \overrightarrow{0, 1}$. Draw the graph, being sure to label the axes.

It will be possible to give a more adequate treatment of formulas in Unit 5, after students have become acquainted with the function concept. In Unit 5 we shall interpret the letters 'c' and 't' in the formula '$c = 2.00 + 3.20t$' as names of <u>functions</u> whose domain is the set of possible repair jobs. Then, the formula is a short way of saying that, for each job J, the charge for J [$c(J)$] is 2.00 plus 3.20 times the time for J [$3.20 \cdot t(J)$]. In other words, the formula is an abbreviation for:

$$\forall_J\ c(J) = 2.00 + 3.20 \cdot t(J).$$

With this interpretation we can speak of the <u>range</u> of the <u>function</u> c, that is, the set of values of this function, rather than, as in the present interpretation, of the <u>domain</u> of the <u>pronumeral</u> 'c'. The range of the function c is the set of numbers of arithmetic which are ≥ 2 [or, with a more strict interpretation of 'repair job', the set of numbers of arithmetic which are > 2].

In view of this forthcoming treatment of formulas, you need not put much stress here on the problem of the domains of the pronumerals. It is sufficient to bring out that the result of a measurement is a number of arithmetic, and that the result of a count is a whole number of arithmetic.

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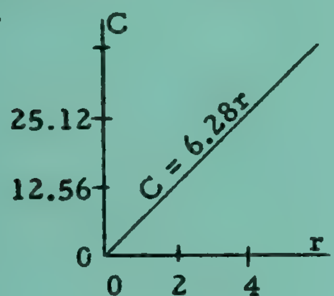
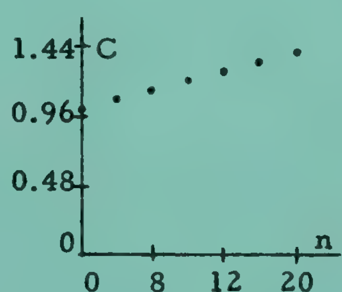
Answers for Exercises.

1.



2. A formula is: $C = 1.00 + .02n$.

   The domain of 'n' is the set of whole numbers of arithmetic. In using the formula, we pretend that the domain of 'c' and of 'n' is the set of real numbers but pay attention only to those results in which the value of 'n' is a nonnegative integer. Thus, the graph consists of discrete dots rather than a streak.



3. (a) \$2.00 (b) \$5.20

   (c) \$8.40 (d) \$3.20

   (e) $c = 2.00 + 3.20t$; the domain of 'c' and of 't' is the set of numbers of arithmetic.

V

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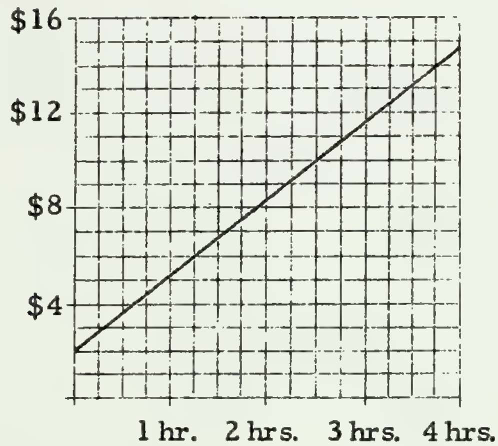
2. A towel service charges $1.00 minimum per week for making two calls and for the use of a container. In addition to the minimum, it charges 2 cents for each clean towel supplied. Write a formula for finding the weekly charge (c) in terms of the number (n) of towels used. The domain of 'c' is the set of numbers of arithmetic, and the domain of 'n' is the set of whole numbers of arithmetic. Make a graph for this formula.

3. A repairman uses the following chart to determine charges for his services.



(a) How much does he charge if he just makes the call and doesn't count any time at all?

(b) How much does he charge if he works one hour?

(c) How much does he charge if he works two hours?

(d) Not counting the charge for making the call, what is his charge per hour for labor?

(e) Give a formula which corresponds to the graph. Tell the domains of the pronumerals in the formula.

(continued on next page)

4. Temperature may be measured on a Fahrenheit scale or on a centigrade scale. Thus, the temperature of boiling water may be given as 212°F or 100°C. A formula for finding the Fahrenheit reading when you know the centigrade reading is:

$$F = \frac{9}{5}C + 32.$$

Make a graph for this formula, and answer the following questions. [What is the domain of 'F' and of 'C'?]

(a) What Fahrenheit reading corresponds to a centigrade reading of 0°?

(b) If the Fahrenheit reading is 80°, what is the centigrade reading?

(c) If the centigrade reading is −15°, what is the Fahrenheit reading?

(d) If there is a 10-degree increase in the temperature measured on the centigrade scale, what is the corresponding increase measured on the Fahrenheit scale?

(e) If there is a 20-degree decrease on the Fahrenheit scale, what is the corresponding decrease on the centigrade scale?

(f) Suppose the out-of-doors temperature rises during the period 12:00 noon to 1:00 p.m. on a certain day. Which of the two scales will show a greater change in readings?

5. The postage charge on first class mail is "4 cents for each ounce or part of an ounce." Use the chart on the next page to find the postage charge for

(a) $2\frac{1}{2}$ ounces (b) $\frac{1}{8}$ ounce

(c) $5\frac{1}{4}$ ounces (d) $\frac{1}{10}$ ounce

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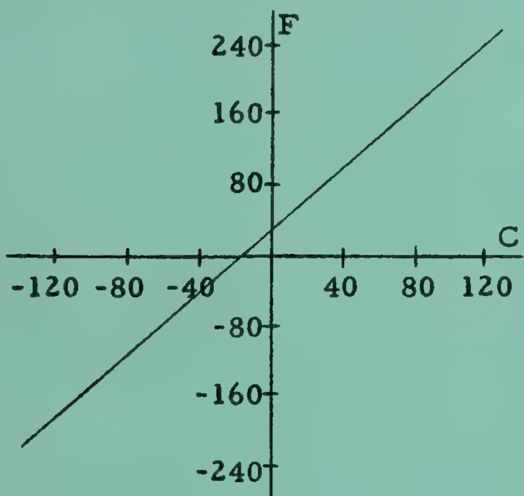
4. [Fundamentally, the results of measuring temperature are numbers of arithmetic--absolute temperatures. But, Fahrenheit and centigrade readings give real numbers which measure increases or decreases of temperature from an arbitrarily chosen one [in the case of centigrade readings, that of melting ice]. You can think, in this connection, of measures of trips from a fixed starting point, using positive numbers to measure trips whose ending point is a higher temperature than the starting point and negative numbers to measure trips whose ending point is a lower temperature than the starting point.]



(a) 32° (b) about 27°

(c) 5° (d) 18°

(e) about 11° (f) Fahrenheit

*

In Exercise 4, there are other questions worth considering. For example:

(g) What values of 'F' correspond with values of 'C' between −100 and 100? [Answer: $\{x: -148 < x < 212\}$.]

(h) Water freezes at 32°F. What is the freezing point of water in degrees centigrade? [Answer: 0°C]

(i) For what temperature are the readings on both scales "the same"? [Answer: −40°C, or: −40°F]

*

5. (a) 12 cents (b) 4 cents (c) 24 cents (d) 4 cents

The domain of 'P' and 'W' is the set of numbers of arithmetic.

4.

[4-41]

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Review Quiz.

1. The graph of (2, 7) is a point in the graph of which of these sets?
   (a) $\{(a, b): 3a + 2b = 25\}$ (b) $\{(m, n): 2m + 3n = 25\}$
   (c) $\{(r, s): 3r - 2s = 25\}$ (d) $\{(x, y): 2x - 3y = 25\}$
   (e) $\{(d, e): -2d + 3e = -25\}$

2. The solution set in (c, d) of which of the following does not contain the origin?
   (a) $5c + 3d = 0$ (b) $3cc + 2d = 8cd$
   (c) $d = -3c$ (d) $1.5c + 2.5d = \frac{32cd}{8}$
   (e) $c + 5 = 3d + 4$

3. There are two numbers whose average is 16. One of the numbers is -65. What is the other number?

4. The average of five scores is 70 less than 3 times their sum. What is their sum?

5. The locus of $\{(a, b): a = 9 \text{ and } b = -5\}$ consists of
   (a) one line only (b) two lines only (c) two points only
   (d) one half-line (e) one point only

6. The second satellite launched by the U.S.S.R. was reportedly 500 miles farther out in space than the first. If these satellites had traveled in circular orbits, the orbit of the second would have been how much longer than that of the first?

*

Answers for Quiz.

1. $\{(m, n): 2m + 3n = 25\}$ 2. $c + 5 = 3d + 4$
3. 97 4. 25 5. one point only
6. 1000π miles [If the orbits were circular, the difference in their length-measures would be $2\pi(r + 500) - 2\pi r$.]

[4-41]

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4.

arbitrarily]. Then, the student thinks: "The first component in the ordered pair tells me the number of cows the farmer might own; therefore, I know that in this case he owns 10 cows and therefore, I know that the cows contribute 40 to the total number of legs. Since there are 570 legs in all, I know that the chickens must contribute 530 legs, and it takes 265 chickens to contribute 530 legs. Therefore, the second component of my ordered pair is 265." The student should find that the largest possible number of cows is 142 because this many cows contribute 568 legs. Therefore, the smallest possible number of chickens is 1. On the other hand, the smallest possible number of cows is 0, because there could be 285 chickens and they would contribute the necessary 570 legs. Could there be exactly 2 chickens? No, because this would mean that the cows would contribute 566 legs, and 566 is not exactly divisible by 4.]

(a) The smallest first component [number of cows] is 0. [Students in some of our pilot schools have argued that since the problem states that the farmer keeps cows <u>and</u> chickens, he must have at least 1 cow! If the class agrees on this interpretation then the answer to (a) is '1'.]

(b) The smallest second component [number of chickens] is 1. [Since there are 570 legs in all, and 142 cows would contribute only 568 legs, the farmer must have at least 1 chicken.]

(c) The largest first component is 142.

(d) The largest second component is 285 [if we assume that the farmer can have 0 cows].

(e) Use 'c' as a pronumeral for numbers of cows, and 'n' as a pronumeral for numbers of chickens. Then, a formula is '$c = \frac{570 - 2n}{4}$'. [The domain of 'c' and of 'n' is the set of whole numbers of arithmetic.]

*

There are endless variations on this problem. You can consider a room in which there are three-legged stools and four-legged chairs and tables with six legs. You can give the total number of furniture legs in the room and begin on a long list of questions as in Ex. 8. [You would need to consider ordered triples instead of ordered pairs.] Or, you can talk about a printed page on which the only kinds of words are words with 1 or 3 or 7 letters and give the total number of letters on a page.

You will recognize that the equations which arise from problems such as these are called <u>linear Diophantine equations</u>. Many of the mathematical riddles and puzzles of folklore are linear Diophantine problems. See Chapter 6 of Ore's <u>Number Theory and Its History</u> (New York: McGraw-Hill, 1948).

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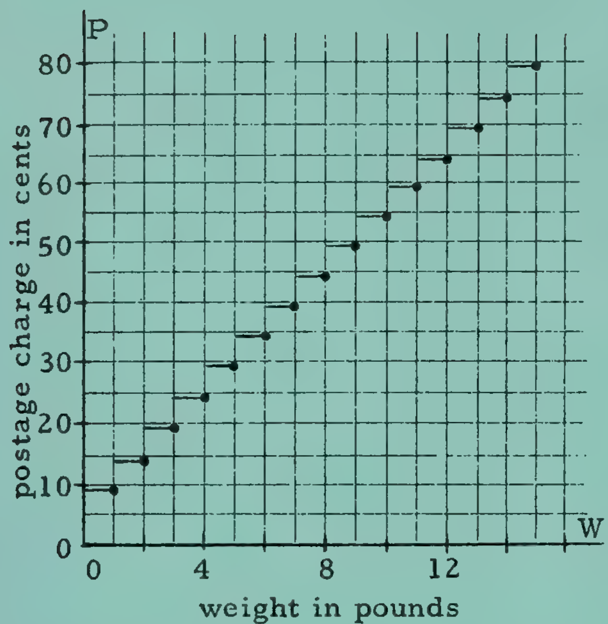
4.

6.



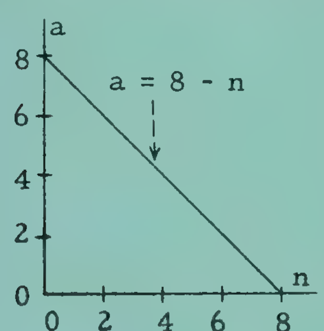
(a) 64 cents

(b) Since there is no smallest number which is greater than 4, there is no minimum weight for a package of books which requires exactly 29 cents postage; the maximum weight of a package which requires exactly 29 cents postage is 5 pounds.

(c) There is no weight for which the postage charge would be exactly 22 cents. So, in this case, there is neither a minimum nor a maximum weight.

(d) The essential difference between this problem and (c) is the phrase 'could be sent with' in contrast with 'require exactly'. We are here interested in weights of packages which can be mailed for at most 27 cents postage. There is no minimum weight of such packages; the maximum weight is 4 pounds. [If you took a 4-pound package and 27 cents to the Post Office, you would have 3 cents left after mailing the package. But, 27 cents in stamps could not be used to mail a package heavier than 4 pounds.]

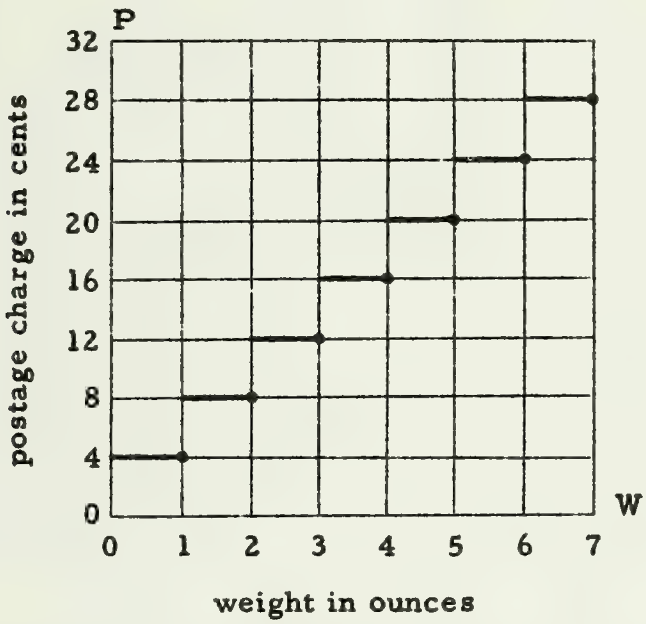
7. One formula is 'a = 8 - n'. The domain of 'a' and of 'n' is the set of numbers of arithmetic. Here is a graph for the formula.



[In Exercise 7, the class may insist that the appropriate kind of numbers to use are the whole numbers of arithmetic. In this case, the graph is a discrete set of dots [as in Exercise 2], rather than a streak as we have pictured, and the domain of 'a' and of 'n' is the set of whole numbers of arithmetic.]

8. [This exercise may seem baffling to the student because of its wordiness. Be sure that the student understands how the sample ordered pair (10, 265) is obtained. The 10 is chosen arbitrarily [or almost

[<u>Note</u>: The graph shown in this chart is often called <u>a</u> <u>step-graph</u>.]



What is the domain of each of the pronumerals 'P' and 'W'?

6. The U.S. Post Office has a "book rate" of 9 cents for the first pound and 5 cents for each additional pound or part of a pound. Make a chart for determining the postage charge for all book shipments which do not exceed 15 pounds in weight.

(a) How much is the postage charge for a 12-pound package of books?

(b) What is the minimum weight and what is the maximum weight of packages which require exactly 29 cents postage?

(c) What is the minimum weight and what is the maximum weight of packages which require exactly 22 cents postage?

(d) What is the minimum weight and what is the maximum weight of packages which could be sent with 27 cents postage?

(continued on next page)

7. The sum of the numbers of years in the ages of two children is 8. Give a formula for finding the age of one child when you know the age of the other. Tell the domain of each of your pronumerals. Make a graph for this formula.

8. A farmer keeps cows and chickens. The total number of legs of these animals is 570. Consider all the ordered pairs in which the first number is a number of cows the farmer might own and the second number is the corresponding number of chickens he would own. For example, one ordered pair is (10, 265). If you listed all such possible ordered pairs, what would be the smallest first component? The smallest second component? The largest first component? The largest second component? Give a formula for finding the number of cows when you know the number of chickens.

4.04 Factors.--In Unit 3 we talked about factoring expressions. Transforming the expression '$x^2 + 6x + 8$' into '$(x + 4)(x + 2)$' is an example of factoring '$x^2 + 6x + 8$'. Each of the expressions '$x + 4$' and '$x + 2$' is a factor of '$x^2 + 6x + 8$'. So, factors of expressions are expressions. In this sense, '3' and '7' are factors of '21' because '21' and '$3 \cdot 7$' are equivalent expressions.

Another use of the noun 'factor' is illustrated by saying that each of the numbers 3 and 7 is a factor of the number 21. Is 5 a factor of 20? Is 1 a factor of 13? Is 3 a factor of 11? Is 6 a factor of 8? Is 10 a factor of 0? Is 0 a factor of 2? In answering these questions you probably considered the facts that

$$20 = 5 \cdot 4,\ \ 13 = 1 \cdot 13,\ \ 11 = 3 \cdot \frac{11}{3},\ \ 8 = 6 \cdot \frac{4}{3},\ \ 0 = 10 \cdot 0,$$

and that, for each x, $0 \cdot x \neq 2$. It may be easy to get everyone to agree that 5 is a factor of 20, that 1 is a factor of 13, that 10 is a factor of 0, and that 0 is not a factor of 2. But, there may be some doubt about whether 3 is a factor of 11 and whether 6 is a factor of 8. There is a number [4] whose product with 5 is 20, there is a number [13] whose product with 1 is 13 and there is a number [$\frac{11}{3}$] whose product with 3 is 11. But, this last case differs from the preceding two in that 4 and 13 are integers but $\frac{11}{3}$ is not. Most people, when they speak of a factor

In this section we deal with the problem of factoring, a problem which is a sticky one. Part of the stickiness can be attributed to the fact that the word 'factor' is used in two primary senses. One speaks both of factors of expressions and factors of numbers. [Words such as 'product', 'sum', 'difference', etc. are also often used in two ways. Up to now, we have tried to use them only in referring to operations with numbers. But, one often does speak of the product of two expressions, etc. By now, you and your students should be able to live with this ambiguity.] What we do in this section is to make precise that use of the word 'factor' which refers to numbers. [The use of 'factor' which refers to expressions is somewhat more complicated. This use is introduced on pages 3-90ff., and is discussed further on pages 4-75ff.]

You will see the fundamental issue if you ask yourself: Is 7 the product of a first number by a second number?, and realize that whether the answer is 'yes' or 'no' depends upon what kind of number you are thinking of when you say 'first number' and 'second number'. If you are referring to real numbers, the answer is 'yes' [7×1, $(-7/2) \times -2$, $\sqrt{7} \times \sqrt{7}$, $\pi \times (7/\pi)$, etc.]. If you are referring to positive integers greater than 1, the answer is 'no'. So, we say that 7 can be factored with respect to the set of real numbers [because, for example, $7 = \pi \times (7/\pi)$, and 7, π, and $7/\pi$ are real numbers], that 7 can be factored with respect to the set of rational numbers [because $7 = (14/3) \times (3/2)$, and 7, 14/3, and 3/2 are rational numbers], and that 7 can be factored with respect to the set of positive integers [because $7 = 7 \times 1$, and 7 and 1 are positive integers]. But, we say that 7 cannot be factored with respect to the set of positive integers greater than 1 because there are no positive integers greater than 1 whose product is the positive integer 7.

In general, it is ambiguous to ask whether a number can be factored. But, given a set of numbers which is closed under multiplication, one can sensibly ask whether one of its members can be factored with respect to this set.

*

In discussing rational numbers, avoid saying that a rational real number is one which can be <u>expressed as</u> the quotient of real integers. A rational real number is a real number which <u>is</u> the quotient of real integers. The phrase 'expressed as' might be used in referring to numerals, but is inappropriate and misleading when one is referring to numbers.

[4-42]

of an integer [such as 20, 13, and 11], mean an integer whose product by some <u>integer</u> is the given integer. On the other hand, it may sometimes be convenient to speak of 3 as a factor of 11. So, we need to give a careful definition of 'factors of a number' which will allow us to treat both the situation in which we want 3 to be a factor of 11 and the situation in which we do not want 3 to be a factor of 11. [Believe it or not, this can be done!]

SUBSETS OF THE SET OF REAL NUMBERS

In our definition we shall need to refer to subsets of the set of real numbers. In particular, we shall consider these subsets:

(1) <u>The set of positive integers</u>
These are the real numbers 1, 2, 3, ... A more careful description of this set is that it is the set of real numbers consisting of 1 together with the numbers obtainable by successive additions of 1.

(2) <u>The set of integers</u>
This is the union of 3 sets--$\{0\}$, the set of positive integers, and the set whose members are the opposites of the positive integers.

(3) <u>The set of rationals</u>
A rational real number is a real number which is the quotient of real integers. For example, $\frac{3}{4}$ is a rational real number because it is the quotient of the integer 3 by the integer 4. Also, -5 is a rational real number because it is the quotient of 5 by -1. [Do you see that every real integer is a rational real number?] The set of rationals is the set consisting of all the rational real numbers. Is 3.7 a rational? Why? Is $-1\frac{5}{8}$ a rational?

[<u>Note</u>: Not every real number <u>is</u> rational. For example, $\sqrt{10}$ is not rational. Neither is π. Each real number which is not a rational is said to be an irrational real number. A <u>ration</u>al real number is the <u>ratio</u> of a real integer to a real integer; an <u>irration</u>al real number is <u>not the ratio</u> of a real integer to a real integer.]

- real numbers
  - rationals
    - integers
      - positive integers
      - 0
      - negative integers
    - nonintegral rationals
  - irrationals

EXERCISES

<u>A</u>. The numbers named below belong to at least one of these sets:

(I) the set of real numbers
(II) the set of rationals
(III) the set of integers
(IV) the set of positive integers

and you are to tell all the sets to which each number listed belongs.

<u>Sample</u>. $^{-}3$

<u>Solution</u>. $^{-}3$ is an integer but not a positive integer. So, $^{-}3$ belongs to set III but not to set IV. Each integer is a quotient of integers, that is, is a rational number. So, $^{-}3$ belongs to set II. Each rational number is a real number, so $^{-}3$ belongs to set I.

<u>Answer</u>. I, II, III.

1. -2	2. $75 + 34$	3. $\sqrt{100}$
4. $2\frac{7}{8}$	5. -3.125	6. $\frac{2}{9}$
7. $-(5/3)$	8. $47 \cdot -13$	9. $(-7)^2$
10. 0	11. $19 - 28$	12. $16 - 35.2$
13 $\frac{17}{19} \cdot \frac{18}{53}$	14. $0.1666...$	15. $0.8333...$

Answers for Part A.

1. I, II, III	2. I, II, III, IV	3. I, II, III, IV	4. I, II
5. I, II	6. I, II	7. I, II	8. I, II, III
9. I, II, III, IV		10. I, II, III	11. I, II, III
12. I, II	13. I, II	14. I, II	15. I, II

[Exercises 14 and 15 lead to discussion introductory to reading page 4-45.]

*

Answers for Part B [on page 4-46].

1. $0.\overline{6}$ 2. $0.\overline{1}$ 3. $0.\overline{27}$ 4. $1.\overline{27}$ 5. $0.\overline{285714}$

*

Answers for Part C [on page 4-46].

[We give fraction-names for the numbers listed.]

1. 1/3	2. 8/9	3. 10/99	4. 307/99
5. 1/2	6. 49/99	7. 361/110	8. 135577/4950

*

Before students can prove that the set of positive integers is closed under [or: with respect to] addition and multiplication, they need a more explicit definition of the set of positive integers than that given on page 4-43. They also need an understanding of the principle of mathematical induction for positive integers. These matters will be taken up in a later unit. However, assuming that

(1) the set of positive integers is closed under addition and multiplication, and

(2) for each two positive integers x and y, either x - y or y - x is a positive integer,

it is possible, using the principles for real numbers and the definition on page 4-43 of the set of integers, to derive the closure properties of this set.

[4-44]

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Answers for Part D [on page 4-46].

1. The set of positive integers is closed under multiplication, but not under subtraction [5 - 7 is not a positive integer] nor under division [2 ÷ 3 is not a positive integer].

2. For (a), we must show that the opposite of each integer is an integer. There are three cases to consider.

   Suppose x is a positive integer. Then, by (2) on page 4-43, the opposite of x is an integer.

   Suppose x is the opposite of a positive integer. Then, by the theorem 'for each x, $- -x = x$', the opposite of the opposite of this positive integer is the positive integer. So, the opposite of x is an integer.

   Finally, $-0 = 0$ [since $0 + 0 = 0$].

   For (b), we must consider several cases.

   First, it follows from the pa0 and the cpa that, for each integer x, $x + 0$ is an integer and $0 + x$ is an integer.

   Second, by assumption (1), for each positive integer x and each positive integer y, $x + y$ is a positive integer, and so, an integer.

   Third, for each negative integer x and each negative integer y, $-x$ and $-y$ are positive integers, so, $-x + -y$ is a positive integer, and so, an integer. Hence, $-(-x + -y)$ is an integer. But, $-(-x + -y) = x + y$.

   Fourth, for each negative integer y, $-y$ is a positive integer and, for each positive integer x, either $x = -y$ or $x \neq -y$. If $x = -y$ then, by the cpa and the po, $x + y = 0$, an integer. If $x \neq -y$ then, by assumption (2), either $x - -y$ or $-y - x$ is a positive integer. So, either $x - -y$ is a positive integer or $-(-y - x)$ is a negative integer. But, $x + y = x - -y = -(-y - x)$, so, in either case, $x + y$ is an integer.

   Fifth, for each negative integer x and each positive integer y, $x + y = y + x$, and, as has just been shown, $y + x$ is an integer.

   For (c), it is sufficient to note the theorem '$\forall_x \forall_y\ x - y = x + -y$' and apply the results proved in (a) and (b).

[4-44]

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For (d), consider five cases, as in (b). The first case is settled by the pm0 and the cpm; the second by assumption; the third by the result proved in (a) and the theorem '$\forall_x \forall_y\ xy = -x \cdot -y$'; the fourth by the result proved in (a) and the theorem '$\forall_x \forall_y\ -(xy) = x \cdot -y$'; the fifth by the result proved in the fourth case and the cpm.

3. The set of rational numbers is closed under the operations of addition, subtraction, multiplication, and [one conventionally says] division (except by 0). [The phrase 'division (except by 0)' refers to the fact that there is no operation division which is defined for the set of (all) rational numbers. But, for all rational number values of 'x' and 'y' for which '$x \div y$' has a value, this value is rational.] [The proof that the set of rational numbers is closed under addition, for example, depends on the "adding fractions" theorem and the fact that the set of integers is closed under addition and multiplication.]

[4-44]

In Part A you may have had some doubts about whether 0.1666... and 0.8333... are rational numbers. But, do you recall that when you try to find the "decimal equivalent" for 1/6 by dividing 1 by 6, your answer doesn't come out "even"?

$$\begin{array}{r} 0.1666 \\ 6\,\overline{)\,1.0000} \\ \underline{6} \\ 40 \\ \underline{36} \\ 40 \\ \underline{36} \\ 40 \end{array}$$

Instead, you keep getting '6's in the quotient numeral. Because of this, we use the expression '0.1666...' [called a repeating decimal] as a name for the rational number 1/6. Similarly, the repeating decimal '0.8333...' is a name for the rational number 5/6.

A common abbreviation for the repeating decimal '0.1666...' is '$0.1\overline{6}$' and for '0.8333...' is '$0.8\overline{3}$'.

Here is another example of a repeating decimal:

$$0.583583583...,$$

or, for short:

$$0.\overline{583}\ .$$

Is $0.\overline{583}$ a rational number? One way to find out is to hunt around for a division problem in which the quotient numeral comes out to be '0.583583583...'. [You might hunt for a long time!] Here is a way to find two integers whose quotient is $0.\overline{583}$ without hunting.

Notice that

$$\begin{aligned} 0.583583583... \times 1000 &= 583.583583583... \\ &= 583 + 0.583583583...\ . \end{aligned}$$

So, 0.583583583... is a solution of the equation:

$$1000k = 583 + k.$$

Since the only solution of this equation is 583/999, it must be the case that

$$0.\overline{583} = \frac{583}{999}\,.$$

[Check by dividing 583 by 999.]

Is $\overline{5.1}$ a rational number?

$$\begin{array}{rcl} 5.15151\ldots \times 100 &=& 515.15151\ldots \\ \underline{5.15151\ldots \times 1} &=& \underline{5.15151\ldots} \\ 5.15151\ldots \times 99 &=& 510. \end{array}$$

So, by the division theorem, $\overline{5.1} = \frac{510}{99}$.

* * *

<u>B</u>. Find decimal names for the rational numbers listed.

1. 2/3 2. 1/9 3. 3/11 4. 14/11 5. 2/7

<u>C</u>. Show that these repeating decimals stand for rational numbers.

1. $0.\overline{3}$ 2. $0.\overline{8}$ 3. $0.\overline{10}$ 4. $3.\overline{10}$

5. $0.4\overline{9}$ 6. $0.\overline{49}$ 7. $3.2\overline{81}$ 8. $27.38\overline{92}$

<u>D</u>. Is each sum of a positive integer and a positive integer also a positive integer? The answer is 'yes' [although, as yet, we have no way of proving this]. For this reason we say that the set of positive integers is <u>closed</u> <u>under</u> <u>addition</u>.

1. Do you think the set of positive integers is closed under multiplication? Under subtraction? Under division?

2. Assume that the set of positive integers is closed under addition and multiplication and that, for each <u>two</u> positive integers x and y, either x - y or y - x is a positive integer. Prove that the set of integers is closed under (a) opposition, (b) addition, (c) subtraction, and (d) multiplication.

3. Under what operations is the set of rational numbers closed?

IRRATIONAL NUMBERS

You have seen how to show that any number which is named by a repeating decimal is a rational number. It is also the case that each rational number can be named by a repeating decimal. [Some rational numbers have two such repeating decimal names. For example, $1/2 = 0.4\overline{9}$ and $1/2 = 0.5\overline{0}$. Also, $74 = 73.\overline{9}$ and $74 = 74.\overline{0}$.]

[4-47]

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5, we may replace '$2 < \sqrt{8} < 3$' by either '$4 \le \sqrt{16} < 5$' or '$3 < \sqrt{16} \le 4$', or '$3 < \sqrt{16} < 5$'. The next three lines then become one of the following columns.

$4 \le p/q < 5$	$3 < p/q \le 4$	$3 < p/q < 5$
$4q \le p < 5q$	$3q < p \le 4q$	$3q < p < 5q$
$0 \le p - 4q < q$	$0 < p - 3q \le q$	$0 < p - 3q < 2q$

But, in the first case we cannot conclude that $p - 4q$ is positive, and in the second and third cases we cannot conclude that $p - 3q$ is smaller than q. In fact, the smallest positive integer q such that, for some integer p, $\sqrt{16} = p/q$ is 1, and the corresponding integer p is 4. So, $p - 4q = 4 - 4 \cdot 1 \not> 0$, and $p - 3q = 4 - 3 \cdot 1 \not< 1$.

☆(14) If $\sqrt{n}$ is rational then there is a smallest positive integer q such that, for some integer p, $\sqrt{n} = p/q$. If $\sqrt{n}$ is not an integer then there is an integer k such that $k < \sqrt{n} < k + 1$. So,

$$k < p/q < k + 1, \qquad kq < p < kq + q, \qquad \text{and} \quad 0 < p - kq < q.$$

Now, $p - kq$ is a positive integer smaller than q. So, since q is the smallest positive integer such that $\sqrt{n} \cdot q$ is an integer,

(*) $\sqrt{n}(p - kq)$ is not an integer.

But, $\sqrt{n} = p/q$. So, $\sqrt{n}(p - kq) = (p/q)(p - kq) = nq - kp$.

Since n, q, k, and p are integers, $nq - kp$ is an integer. That is

(**) $\sqrt{n}(p - kq)$ is an integer.

(**) contradicts (*). Since (*) and (**) follow from the suppositions that $\sqrt{n}$ is rational, and that $\sqrt{n}$ is not an integer, this contradiction shows that there is no positive integer n such that $\sqrt{n}$ is not an integer and $\sqrt{n}$ is rational. So, for each positive integer n, if $\sqrt{n}$ is not an integer then $\sqrt{n}$ is not rational.

☆(15) If $\sqrt{n - 1} + \sqrt{n + 1}$ is rational, then so is $(\sqrt{n - 1} + \sqrt{n + 1})^2$. That is, $2n + 2\sqrt{n^2 - 1}$ is rational. Since [if n is a positive integer] 2n is rational, $2\sqrt{n^2 - 1}$ is rational. And, since 1/2 is rational, $\sqrt{n^2 - 1}$ is rational. So, by the theorem of ☆(14), there is an [positive] integer m such that $\sqrt{n^2 - 1} = m$ [that is, $n^2 - 1 = m^2$]. Hence, if n is a positive integer and $\sqrt{n - 1} + \sqrt{n + 1}$ is rational then there is a positive integer m such that $n^2 - m^2 = 1$ [that is, $(n - m)(n + m) = 1$]. Since m is positive, $n - m \ne n + m$. So, it follows that 1 is the product of two integers. But, this is not the case. Consequently, if n is a positive integer then $\sqrt{n - 1} + \sqrt{n + 1}$ is irrational.

[4-47]

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(8) If $\sqrt{2} + \sqrt{3}$ is rational then so is $(\sqrt{2} + \sqrt{3})^2$--that is, $5 + 2\sqrt{6}$ is rational. Since 5 is rational and the set of rational numbers is closed under subtraction [$5 + 2\sqrt{6} - 5 = 2\sqrt{6}$], it follows that $2\sqrt{6}$ is rational. So, $\sqrt{6}$ is rational. But, this is not the case [$\sqrt{6}$ is irrational; see ☆(14) on page 4-48]. Hence, $\sqrt{2} + \sqrt{3}$ is not rational.

(9) Yes, because, for each $x \geq 0$, $x = (\sqrt{x})^2$, and the set of integers is closed under multiplication.

(10) Yes. This is equivalent to the question posed in (9).

(11) No. 8 is an integer, but $\sqrt{8}$ is not [See starred section immediately following question (11).].

*

The proof, given on page 4-48, that $\sqrt{8}$ is not rational is at least as simple as the conventional proof [see Exercise ☆8 of Part A on page 4-55] that $\sqrt{2}$ is irrational. The proof given here has the advantage that it is readily generalized to a proof of the theorem of ☆(14) on page 4-48. Students should become acquainted with this latter theorem, even if they do not prove it. For example, they should see that from this theorem and the fact that $\sqrt{6}$ is not an integer it follows that $\sqrt{6}$ is irrational. [That $\sqrt{6}$ is not an integer follows from the fact that $\sqrt{6} \geq 0$ and that the square of each nonnegative integer not greater than 2 is less than 6, while the square of each integer greater than 2 is greater than 6.]

*

The final sentence of the first paragraph on page 4-48 refers to the fact that each nonempty subset of the set of positive integers contains a smallest member. Students should accept this as intuitively clear. An intuitively convincing remark is that if you choose some member of a given set of positive integers then, if it is not the smallest member of this set, you can "count back" through the members of the set until you reach the smallest one. [Clearly, this is not a property of the real numbers.]

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☆(12) [Repeat the preceding discussion on page 4-48 with '8' replaced by '31', '2' replaced by '5', and '3' replaced by '6'.]

☆(13) Short answer: Since $\sqrt{16}$ is an integer, we cannot find <u>consecutive</u> integers which "bracket" $\sqrt{16}$.

Expansion of short answer: There are three fairly obvious ways to modify the proof given on page 4-48 so that it will apply to $\sqrt{16}$. [Of course, in any case, we replace '8' by '16'.] Starting with line

[4-47]

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By 'decimal name' in the first line of page 4-47, we mean nonterminating decimal name.

*

Answers for problems on page 4-47.

(1) If $3\sqrt{5}$ were rational then $3\sqrt{5} \div 3$ would be a rational number because 3 is a nonzero rational number and the quotient of any rational number by any nonzero rational number is rational. But, $3\sqrt{5} \div 3 = \sqrt{5}$. So, if $3\sqrt{5}$ were rational, then $\sqrt{5}$ would be rational--that is, if $\sqrt{5}$ is irrational then so is $3\sqrt{5}$.

(2) [As in (1)], for each r, for each x, if r is a nonzero rational number and rx is a rational number then $rx \div r$ [that is, x] is a rational number. So, if r is a nonzero rational number and x is an irrational number, it is not the case that rx is a rational number. [Of, course, for <u>each</u> x, $0 \cdot x = 0$, a rational number.]

(3) If $\pi + 5$ were rational then $\pi + 5 - 5$ would be a rational number because 5 is a rational number and the set of rational numbers is closed under subtraction. But, $\pi + 5 - 5 = \pi$. So, if $\pi + 5$ were rational, then π would be rational--that is, if π is irrational then so is $\pi + 5$.

(4) [As in (3)], for each r, for each x, if r is rational and $x + r$ is rational, then $x + r - r$ [that is, x] is rational. So, if r is rational and x is irrational then $x + r$ is irrational.

(5) Since $\sqrt{11} \neq 0$, and since, for each $x \neq 0$, $1/x \neq 0$, $1/\sqrt{11} \neq 0$. So, if $1/\sqrt{11}$ is rational it is a nonzero rational number, and if $\sqrt{11}$ is irrational then, by the result proved in (2), $(1/\sqrt{11})\sqrt{11}$ is irrational. But, $(1/\sqrt{11})\sqrt{11} = 1$, and 1 is a rational number. Hence, if $\sqrt{11}$ is irrational then it is not the case that $1/\sqrt{11}$ is rational.

(6) [As in (5)], since, for each irrational number x, $x \neq 0$ [because 0 is a rational number] and since, for each $x \neq 0$, $1/x \neq 0$, it follows that, for each irrational number x, $1/x$ is a nonzero real number. So, for each irrational number x, if $1/x$ is a rational number then $1/x$ is a nonzero rational number and, by the result proved in (2), $(1/x)x$ [that is, 1] is an irrational number. But, 1 is rational. Hence, if x is irrational then so is $1/x$.

(7) No. π and $-\pi$ are irrational numbers, but $\pi + -\pi = 0$; π and $1/\pi$ are irrational numbers, but $\pi(1/\pi) = 1$.

So, any real number whose decimal name is not a repeating decimal is an irrational number. For example,

1.01001000100001... is an irrational number.

More interesting examples of irrational numbers are

$\sqrt{2}$, $\sqrt{3}$, π, $\sqrt{131}$, $5\sqrt{5}$, and $1/\sqrt{3}$.

Suppose you know that $\sqrt{7}$ is irrational. Does it follow that $\sqrt{7}/3$ is irrational? Yes, because if $\sqrt{7}/3$ were rational then $(\sqrt{7}/3) \cdot 3$, that is, $\sqrt{7}$, would be a product of rational numbers. And, you know that each product of rationals is rational.

(1) Given that $\sqrt{5}$ is irrational, show that $3\sqrt{5}$ is irrational.

(2) Is each product of a rational number by an irrational number an irrational number?

(3) Given that π is irrational, show that $\pi + 5$ is irrational.

(4) Is each sum of an irrational number and a rational number irrational?

(5) Given that $\sqrt{11}$ is irrational, show that $1/\sqrt{11}$ is irrational.

(6) Is the reciprocal of each irrational number irrational?

(7) Is each sum [product] of two irrational numbers irrational?

(8) Show that $\sqrt{2} + \sqrt{3}$ is irrational. [Hint: Square the number.]

(9) Is it the case that, for each $x \geq 0$, if $\sqrt{x}$ is an integer then so is x?

(10) Is it the case that, for each $x \geq 0$, if x is not an integer then $\sqrt{x}$ is not an integer?

(11) Is it the case that, for each $x \geq 0$, if x is an integer then so is $\sqrt{x}$?

*

☆ The answer to the last question is 'no' as is shown by the fact that 8 is a counter-example. 8 is an integer but, since $2^2 < 8 < 3^2$ and there is no integer between 2 and 3, $\sqrt{8}$ is not an integer. As a matter of fact, $\sqrt{8}$ is not even a rational number. Let's prove this.

Suppose $\sqrt{8}$ were rational. Then there would be many positive integers which could be divided into integers to give $\sqrt{8}$. Among these positive integers there would have to be a smallest one.

Let q be the smallest positive integer such that, for some integer p, $\sqrt{8} = p/q$. Since $2 < \sqrt{8} < 3$,

$$2 < \frac{p}{q} < 3,$$

$$2q < p < 3q,$$

and $$0 < p - 2q < q.$$

Now, $p - 2q$ is a positive integer <u>smaller</u> than q. So, since q is the smallest positive integer such that $\sqrt{8} \cdot q$ is an integer,

(*) $\sqrt{8}(p - 2q)$ is not an integer.

But, $\sqrt{8} = p/q$. So,

$$\begin{aligned} &\sqrt{8}(p - 2q) \\ &= \frac{p}{q}(p - 2q) \\ &= \frac{p^2}{q} - 2p \\ &= \frac{p^2}{q^2}q - 2p \\ &= 8q - 2p. \end{aligned}$$

Since q and p are integers, $8q - 2p$ is an integer. That is,

(**) $\sqrt{8}(p - 2q)$ is an integer.

(**) contradicts (*). Since (*) and (**) follow from the supposition that $\sqrt{8}$ is rational, this contradiction shows that $\sqrt{8}$ is <u>not</u> rational.

*

☆(12) Prove that $\sqrt{31}$ is irrational.

☆(13) What happens when you try to use the method illustrated above to show that $\sqrt{16}$ is irrational?

☆(14) Prove that, for each positive integer n, if $\sqrt{n}$ is not an integer then $\sqrt{n}$ is irrational.

☆(15) Prove that, for each positive integer n, $\sqrt{n - 1} + \sqrt{n + 1}$ is irrational.

ay,

Here are the explanations asked for in lines 10 through 14.

-5 is a factor of 15 with respect to the set of integers because 15 is the product of -5 by -3, and -3 is an integer [as are -5 and 15].

-5 is not a factor of 15 with respect to the set of positive integers because -5 is not a positive integer.

-5 is a factor of 15 with respect to the set of rationals because 15 is the product of -5 by -3, and -3 is a rational [as are -5 and 15]. Another way of seeing this is to recall that -5 is a factor of 15 with respect to the set of integers, and to note that the set of integers is a subset of the set of rationals. In general, if one number is a factor of another with respect to one set of numbers, it is also a factor of this number with respect to each <u>more</u> <u>inclusive</u> set.

*

Answers for Part A [on pages 4-49, 4-50, and 4-51].

[In most of these exercises, it would be correct to fill one of the blanks with the name of any one of several sets. In such cases, we give at least two choices.]

1. positive integers [or: reals, or: rationals], 13, 13
2. integers [or: reals, or: rationals], -4, -4
3. integers [or: reals, or: rationals], 2, 2
4. rationals [or: reals], $\frac{4}{3}$, $\frac{4}{3}$
5. rationals [or: reals], 18, 18
6. rationals [or: reals], $\frac{10}{51}$, $\frac{10}{51}$
7. positive integers [or: reals, or: rationals] 8, 8
8. reals, $\sqrt{10}$, $\sqrt{10}$
9. integers [or: reals, or: rationals] 0, 0
10. 0, integers [or: reals, or: rationals] 0, 0, 0
11. positive integers [or: reals, or: rationals], 429, 429
12. positive integers [or: reals, or: rationals], $89 + 11$, $89 + 11$

FACTORS OF NUMBERS

We mentioned that we want a definition which will allow us to say, sometimes, that 3 is not a factor of 11, and, sometimes, that 3 is a factor of 11. We accomplish this by saying that

> 3 is not a factor of 11 with respect to the set of integers because 11 is not the product of 3 by any integer,

and by saying that

> 3 is a factor of 11 with respect to the set of rationals because 11 is the product of 3 by some rational number [11/3].

Is −5 a factor of 15? This question is ambiguous. −5 is a factor of 15 with respect to the set of integers [Explain], but −5 is not a factor of 15 with respect to the set of positive integers [Explain]. Is −5 a factor of 15 with respect to the set of rationals? With respect to the set of irrationals? With respect to the set of reals?

In general,

for each set S of numbers,

x is a factor of y with respect to S
if and only if
x and y are in S, and there is a z in S
such that y = xz.

EXERCISES

A. Complete each of these sentences to true ones in at least one way.

Sample. 4 is a factor of 7 with respect to the set of ______ because 4, 7, and ____ belong to this set and 7 = 4 · ____.

Solution. One completion:

4 is a factor of 7 with respect to the set of *rationals* because 4, 7, and *7/4* belong to this set and 7 = 4 · *7/4*.

Another completion:

4 is a factor of 7 with respect to the set of *reals* because 4, 7, and *7/4* belong to this set and 7 = 4 · *7/4*.

(continued on next page)

1. 3 is a factor of 39 with respect to the set of __________ because 3, 39, and ____ belong to this set and $39 = 3 \cdot$ ____.

2. 5 is a factor of -20 with respect to the set of __________ because 5, -20, and ____ belong to this set and $-20 = 5 \cdot$ ____.

3. -10 is a factor of -20 with respect to the set of __________ because -10, -20, and ____ belong to this set and $-20 = -10 \cdot$ _____.

4. -6 is a factor of -8 with respect to the set of __________ because -6, -8, and ____ belong to this set and $-8 = -6 \cdot$ _____.

5. $\frac{1}{2}$ is a factor of 9 with respect to the set of __________ because $\frac{1}{2}$, 9, and ____ belong to this set and $9 = \frac{1}{2} \cdot$ _____.

6. $\frac{3}{5}$ is a factor of $\frac{2}{17}$ with respect to the set of __________ because $\frac{3}{5}$, $\frac{2}{17}$, and _____ belong to this set and $\frac{2}{17} = \frac{3}{5} \cdot$ _____.

7. 8 is a factor of 64 with respect to the set of __________ because 8, 64, and ____ belong to this set and $64 = 8 \cdot$ _____.

8. $\sqrt{10}$ is a factor of 10 with respect to the set of __________ because $\sqrt{10}$, 10, and _____ belong to this set and $10 = \sqrt{10} \cdot$ ____.

9. 9 is a factor of 0 with respect to the set of __________ because 9, 0, and ____ belong to this set and $0 = 9 \cdot$ ____.

10. 0 is a factor of ____ with respect to the set of __________ because 0, ____, and 0 belong to this set and ____ $= 0 \cdot$ ____.

11. 783 is a factor of $783 \cdot 429$ with respect to the set of __________ because 783, $783 \cdot 429$, and ________ belong to this set and $783 \cdot 429 = 783 \cdot$ _______.

12. 17 is a factor of $17 \cdot 89 + 17 \cdot 11$ with respect to the set of __________ because 17, $17 \cdot 89 + 17 \cdot 11$, and _______ belong to this set and $17 \cdot 89 + 17 \cdot 11 = 17 \cdot$ ______.

[4-51]

So,

ers.

1. ?

13. Any integer will work. [Of course, certain nonintegral rationals like 2/3 and 5/3 will also work, but keep in mind that we are preparing for Exercise 2 of Part B.]

14. 13

*

Answers for Part B.

1. (a) If a is a factor of b and of c with respect to the set of integers, then there are integers m and n such that b = am and c = an. So,

$$\begin{aligned} b + c &= am + an \\ &= a(m + n). \end{aligned}$$

Since m and n are integers, so is m + n. Hence, a is a factor of b + c with respect to the set of integers.

(b) Yes. [In fact, each similar generalization about a set of numbers which is closed under both addition and multiplication is a theorem.]

2. Generalizations:

$\forall_a \forall_b \forall_c$, with respect to the set of integers,
if a is a factor of b then a is a factor of bc.

Proof: If a is a factor of b with respect to the set of integers then there is an integer m such that b = am. Consequently, bc = amc = a(mc). Since m and c are integers, so is mc. Therefore, a is a factor of bc with respect to the set of integers.

[Note that each similar generalization about a set of numbers which is closed under multiplication is a theorem.]

3. a is a factor of c

4. $\forall_a \forall_b \forall_c$, with respect to the set of integers,
if a is a factor of b and of c then a^2 is a factor of bc.

5. Yes. [This follows from the theorem of the preceding exercise.]

13. 11 is a factor of 33 with respect to the set of integers. So, 11 is a factor of $33 \cdot$ ____ with respect to the set of integers.

14. 13 is a factor of 26 and of 39 with respect to the set of integers. So, ____^{2} is a factor of $26 \cdot 39$.

[Supplementary exercises are in Part I, pages 4-124 and 4-125.]

B. 1. Exercise 12 of Part A suggests an interesting generalization. You noticed there that 17 is a factor of both $17 \cdot 89$ and $17 \cdot 11$ with respect to the set of integers [and perhaps other sets]. From this you probably concluded that 17 is a factor of the sum of $17 \cdot 89$ and $17 \cdot 11$ with respect to the set of integers. In fact, you probably suspect that the following generalization is a theorem:

$\forall_a \forall_b \forall_c$, with respect to the set of integers,

if a is a factor of b and of c then a is a factor of b + c.

(a) Prove it. [Hint: If a is a factor of b and of c with respect to the set of integers, then there are integers m and n such that b = am and c = an. ....]

(b) In the theorem above, replace 'integers' by 'positive integers'. Is the resulting generalization a theorem?

2. Exercise 13 of Part A suggests a generalization about a factor of the product of two integers. State and prove it.

3. Complete this to a theorem equivalent to the one just proved in Exercise 2:

$\forall_a \forall_b \forall_c$, with respect to the set of integers,

if a is a factor of b and b is a factor of c

then ______________________________.

4. State the theorem suggested by Exercise 14 of Part A.

5. Is it the case that, with respect to the set of integers, if a first number is a factor of a second, then the square of the first is a factor of the square of the second?

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EVEN AND ODD NUMBERS

An <u>even number</u> is one which has 2 as a factor with respect to the set of integers.

Is 0 an even number? Is $^-4$? Is $\frac{2}{3}$? Is 3? Is 0.02?

Is the set of even numbers closed under oppositing?

Is the set of even numbers closed under addition? Prove that it is. How about subtraction? Multiplication? Division? Squaring? Square rooting?

Is it the case that each product of an even number by an integer is even? Prove that it is.

*

An <u>odd number</u> is an integer which is not even.

Is 7 an odd number? [From your earlier experience, you know that the answer is 'yes'. But, let's see if this fits the definition we just gave.] According to the definition, to show that the integer 7 is an odd number, we must show that 7 is not even. Let's do so.

If 7 were even then 7/2 would be an integer, and a positive one at that. We described the set of positive integers [page 4-43] as the set of real numbers consisting of 1 together with the real numbers obtainable by successive additions of 1. The first three positive integers are 1, 2, and 3. Since each of these is less than 7/2, 7/2 is not one of them. The next positive integer is 4 and the remaining positive integers are greater than 4. So, since $7/2 < 4$, 7/2 is not one of these. So, 7/2 is not a positive integer, and, therefore, 7 is not even.

This is a lot of work to show that a given integer is odd. There should be a better way. What we need is a theorem.

Given an integer n. Either n/2 is an integer or n/2 is not an integer. n/2 is an integer if and only if n is an even number. So, n is an odd number if and only if n/2 is not an integer. Now, if n/2 is not an integer, there is a pair of consecutive integers which bracket n/2. That is, there is an integer k such that

$$(1) \qquad k < \frac{n}{2} < k + 1.$$

Conversely, if there is such an integer k, n/2 is not an integer.

Answers for questions in lines 4 through 10 on page 4-52.

0 and $^{-}4$ are even numbers; 2/3, 3, and 0.02 are not.

The set of even numbers is closed under oppositing. [For each x, $-(2x) = 2 \cdot -x$, and the set of integers is closed under oppositing.]

The set of even numbers is closed under addition. [This follows from the theorem in Exercise 1 of Part B on page 4-51.]

Since the set of even numbers is closed under both addition and oppositing, it is also closed under subtraction.

That the set of even numbers is closed under multiplication is a consequence of the theorem of Exercise 2 of Part B on page 4-51.

The set of even numbers is not closed under division, for 2 and 4 are even numbers, but $2 \div 4$ is not an integer, still less an even number; also, 6 and 2 are even numbers, but $6 \div 2$, while an integer, is not an even number.

The set of even numbers is closed under squaring because it is closed under multiplication.

The set of even numbers is not closed under square rooting, for 6 is an even number, but $\sqrt{6}$ is not an integer. [But, if the square root of an even number is an integer, this square root must be an even number. See Exercise 7 of Part A on page 4-55.]

That the product of an even number by an integer is even follows from the theorem in Exercise 2 of Part B on page 4-51. [If b is an even number then 2 is a factor of b with respect to the set of integers. So, if c is an integer, 2 is a factor of bc with respect to the integers. Hence, bc is an even number.]

[4-52]

EVF

Problems like these can be found in any textbook dealing with the elements of the Theory of Numbers. When students reach the factoring exercises which begin on page 4-78, you may want to give them problems such as:

> Prove that, for each integer x, 30 is a factor of $x^5 - x$ with respect to the set of integers.

*

An interesting problem which will be a real challenge for your students is the following:

> Prove that, for each positive integer n, each set of n integers contains a subset such that n is a factor, with respect to the set of all integers, of the sum of the members of the subset.

This theorem tells you, for example, that if you pick any 13 integers, some of them will have a sum which is exactly divisible by 13. [The conclusion of the theorem holds if instead of a set of n integers, one considers a <u>sequence</u> of n integers which are not necessarily distinct. Then, there exists a subsequence whose sum is exactly divisible by n.]

Here is a rationale for the test described in the next-to-last paragraph on page 4-53. If k is the number named by the last digit in the decimal numeral for an integer n then there is an integer m such that $n = 10m + k$. Since $10 = 2 \cdot 5$, 10 is even, and, since the product of an even number by an integer is even, it follows that 10m is even. Now, since the set of even numbers is closed under addition, if k is even then $10m + k$ is even. So, since 0, 2, 4, 6, and 8 are even, if the last digit in the decimal numeral for n is a '0', a '2', a '4', a '6', or an '8', then n is an even number. On the other hand, the number named by any other digit is odd, and it follows from the theorem of the Example on page 4-54 [and the cpa] that the sum of an even number and an odd number is an odd number.

*

Here is an answer to the 'Why?' in the last paragraph on page 4-53. Suppose that m and n are integers, neither of which is even. Then, both m and n are odd, and there are integers x and y such that $m = 2x + 1$ and $n = 2y + 1$. So,

$$\begin{aligned} m + n &= 2x + 1 + (2y + 1) \\ &= 2(x + y + 1). \end{aligned}$$

Since x, y, and 1 are integers, and since the set of integers is closed under addition, $x + y + 1$ is an integer. So, there is an integer k such that $m + n = 2k$. Hence, $m + n$ is even.

Therefore, if m and n are integers, neither of which is even, then $m + n$ is even.

*

Answer to question at bottom of page 4-54: Yes, because if both are even, or both are odd, then their sum is even.

*

Some of your students may be interested in more problems somewhat like those at the bottom of page 4-53 and in the exercises on page 4-55. For example:

> Prove that, with respect to the set of integers, 6 is a factor of the product of each three consecutive integers.

So, (1) is equivalent to:

$$0 < \frac{n}{2} - k < 1,$$

and so to:

$$(2) \qquad 0 < n - 2k < 2.$$

But, since n and 2k are integers, n - 2k is an integer. And, since 1 is the only integer between 0 and 2, sentence (2) is the case if and only if n - 2k = 1, that is, if and only if

$$(3) \qquad n = 2k + 1.$$

So, n is an odd number if and only if there is an integer k such that n = 2k + 1. For example, 7 is odd because $7 = 2 \cdot 3 + 1$. Also, $2 \cdot 9835416 + 1$ is an odd number.

We have proved the following theorem:

$\forall_n$ [n is an odd number
if and only if
there is an integer k
such that n = 2k + 1].

There is a similar theorem for even numbers which follows easily from the definition of 'even number'.

$\forall_n$ [n is an even number
if and only if
there is an integer k
such that n = 2k].

These two theorems are the justification for the quick way of telling when an integer is an odd number--just find out if the integer is 1 more than an even number.

*

There is another way of telling quickly whether an integer is even or odd. Take a look at the standard decimal numeral for the integer. If the last digit is a '0', a '2', a '4', a '6', or an '8', the integer is even. Otherwise, it is odd. Can you explain why this test works?

*

Pick a pair of integers. If neither is even then their sum is even. Why? This question and others like it will be very easy to answer after you read the next page.

The two theorems on page 4-53 about even and odd numbers make it easy to prove other theorems about evenness and oddness.

Example. Prove that each sum of an odd number and an even is odd.

First, let's state the theorem in a form which will help us write the proof:

$$\forall_m \forall_n \text{ if } m \text{ is odd and } n \text{ is even then } m + n \text{ is odd.}$$

We want to start with the premiss:

m is odd and n is even

and derive from it the conclusion:

m + n is odd.

The two theorems we have proved give us "standard forms" for odd and even numbers. For example, to show that m + n is odd, it is sufficient to show that m + n is 1 more than the product of 2 by an integer.

Proof.

Suppose that m is odd and n is even. Then there are integers x and y such that

$$m = 2x + 1 \text{ and } n = 2y.$$

So,
$$\begin{aligned} m + n &= 2x + 1 + 2y \\ &= 2(x + y) + 1. \end{aligned}$$

Since x and y are integers, and since the set of integers is closed under addition, it follows that there is an integer k such that

$$m + n = 2k + 1.$$

So, m + n is odd.

Therefore, if m is odd and n is even then m + n is odd.

Is it the case that if the sum of two integers is odd then one of them is even and the other odd?

[4-55]

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5. 0 is the only one. [Each rational number except 0 is a factor of 12 with respect to the set of rationals.] [Of course, no number other than a rational number <u>can</u> be a factor of 12 with respect to the set of rational numbers.]

   0 is also the only real number which is not a factor of 12 with respect to the set of reals. [If you replace 'rational' in the foregoing remarks by 'real', they are appropriate to the second question.]

6. 1 is the only factor of 1 with respect to the set of positive integers.

7. Suppose n is a positive integer other than 1. Then, by the pml, $n = n \cdot 1$. Since $n \neq 1$, it follows that n and 1 are <u>two</u> factors of n.

8. 4, 9, 25, 49, 121. [The only such numbers are the squares of prime numbers.]

9. 2, 3, 5, 7, 11. [The only such numbers are the prime numbers.]

10. 2, 3, 5, 7, 11. [The prime numbers; a prime number has <u>no</u> factor with respect to the set of <u>positive-integers-greater-than-1</u>.]

[4-55]

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even, $m + 1$ is odd. So, by the Example on page 4-54, and the cpa, $m + (m + 1)$ is odd. And one shows that, in case m is odd, $m + 1$ is even. So, by this same Example, $m + (m + 1)$ is odd.]

6. Suppose that p is an integer and is not odd. Then p is even, and, since the set of even numbers is closed under multiplication, p^2 is even. So, p^2 is not odd. Hence, if p is not odd then p^2 is not odd. So [contrapositive], if p^2 is odd then p is odd.

7. [This proof is obtained by replacing, in the proof for Exercise 6, 'odd' by 'even', and 'even' by 'odd'. That the set of odd numbers is closed under multiplication has been proved in answering Exercise 3.]

☆8. If $\sqrt{2}$ is rational then there are integers x and y such that $\sqrt{2} = x/y$, and such that x and y are not both even. [For, supposing that $\sqrt{2}$ is rational, there are many pairs of integers whose quotient is $\sqrt{2}$, and by "reducing to lowest terms" one can find such a pair which are not both even.] So, $x^2 = 2y^2$. Since y is an integer, and since the set of integers is closed under multiplication, it follows that there is an integer k such that $x^2 = 2k$. So, x^2 is even and, by the theorem of Exercise 7, x is even. Consequently, there is an integer m such that $x = 2m$. So, $x^2 = 4m^2$ and, since $x^2 = 2y^2$, $y^2 = 2m^2$. An argument similar to the above now shows that y is even.

So, if $\sqrt{2}$ is rational then there are integers x and y such that

(a) x and y are not both even, and

(b) x is even and y is even.

Therefore, $\sqrt{2}$ is not rational.

*

Answers for Exploration Exercises [on pages 4-55 and 4-56].

1. 1, 2, 3, 4, 6, 12

2. 1, −1, 2, −2, 3, −3, 4, −4, 6, −6, 12, −12

3. 1, −1, 2, −2, 3, −3, 4, −4, 6, −6, 12, −12

4. There are none; that is, there are no two negative integers whose product is −12. [This shows that the set of negative integers is not closed with respect to multiplication.]

[4-55]

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Answers for Exercises.

1. [The proof asked for is contained in the second COMMENTARY on TC[4-53, 54]a.]

2. [One method for proving this theorem is suggested on TC[4-52]a. We give here another proof like that of the Example on page 4-54.]

   Suppose that m and n are even numbers. Then, there are integers x and y such that m = 2x and n = 2y. So, mn = 2x(2y) = 2(x2y). Since x, 2, and y are integers, and since the set of integers is closed under multiplication, it follows that there is an integer k such that mn = 2k. So, mn is even. Therefore, if m and n are even then mn is even.

3. Suppose that m and n are odd numbers. Then there are integers x and y such that m = 2x + 1 and n = 2y + 1. So, mn = (2x + 1)(2y + 1) = 2(x2y) + 2x + 2y + 1 = 2(x2y + x + y) + 1. Since x, 2, and y are integers, and since the set of integers is closed under both multiplication and addition, it follows that there is an integer k such that mn = 2k + 1. So, mn is odd. Therefore, if m and n are odd then mn is odd.

4. Suppose that m and m + 1 are consecutive integers. Either m is even or m is odd.

   Suppose m is even. Then there is an integer x such that m = 2x. So, m(m + 1) = 2x(m + 1) = 2[x(m + 1)]. Since x and m + 1 are integers, and since the set of integers is closed under multiplication, it follows that there is an integer k such that m(m + 1) = 2k. So, m(m + 1) is even.

   Suppose m is odd. Then there is an integer x such that m = 2x + 1. So, m + 1 = 2x + 1 + 1 = 2(x + 1). Consequently, m(m + 1) = m[2(x + 1)] = 2[m(x + 1)]. Since m, x, and 1 are integers, and since the set of integers is closed under both addition and multiplication, it follows that there is an integer k such that m(m + 1) = 2k. So, m(m + 1) is even.

   Since m is either even or odd, and since in either case, m(m + 1) is even, it follows that m(m + 1) is even.

5. [This proof should cause no difficulty. It is similar in structure to that for the preceding exercise. One notes that, in case m is

EXERCISES

Prove these theorems.

1. Each sum of an odd number and an odd number is even.

2. The set of even numbers is closed under multiplication.

3. Each product of an odd number by an odd number is odd.

4. Each product of two consecutive integers is even.
   [Hint: Suppose m and m + 1 are consecutive integers. Either m is even or m is odd. Suppose m is even. Then ... . Suppose m is odd. Then ... .]

5. Each sum of two consecutive integers is odd.

6. For each integer p, if p^2 is odd then p is odd.
   [Hint: Suppose that p is not odd. From this, what can you say about p? And from that, what can you say about p^2?]

7. For each integer p, if p^2 is even then p is even.

☆8. Use the result of Exercise 7 to prove that $\sqrt{2}$ is irrational.
   [Hint: If $\sqrt{2}$ is rational then there are integers x and y such that $\sqrt{2} = x/y$, and such that x and y are not both even [Why?]. So, $x^2 = 2y^2$ ... . Continue the argument, showing first that x must be even, and then that it follows that y^2 must be even. Complete the proof.]

EXPLORATION EXERCISES

1. What are all the factors of 12 with respect to the set of positive integers?

2. What are all the factors of 12 with respect to the set of integers?

3. What are all the factors of -12 with respect to the set of integers?

4. What are all the factors of -12 with respect to the set of negative integers?

5. What [rational] numbers are not factors of 12 with respect to the set of rationals? The set of reals?

6. What are the factors of 1 with respect to the positive integers?

(continued on next page)

7. Show that each positive integer other than 1 has at least two factors with respect to the set of positive integers.

8. Give five numbers which have exactly three factors with respect to the set of positive integers.

9. Give five numbers which have exactly two factors with respect to the set of positive integers.

10. Give five positive integers other than 1, each of which has no factor with respect to the set of positive-integers-greater-than-1.

PRIME NUMBERS

A positive integer which has exactly two factors with respect to the set of positive integers is a <u>prime</u> <u>number</u>. So, for example, 5 is a prime number since the only positive integers which are factors of 5 with respect to the set of positive integers are 1 and 5. 7 is another prime number. Is 1 a prime number? One factor of 1 with respect to the set of positive integers is 1. Does 1 have <u>another</u> factor with respect to the set of positive integers?

EXERCISES

<u>A</u>. Which of these numbers are prime numbers?

2, 6, 11, 1, 3, 17, 15, 84972, 61

<u>B</u>. 1. Show that a positive integer other than 1 is a prime number if, with respect to the set of positive integers, it has no factors other than itself and 1.

2. Show that a positive integer other than 1 is a prime number if it has no factor with respect to the set of positive-integers-greater-than-1.

You may have seen a definition of 'prime number' according to which a prime number is a positive integer whose only factors are 1 and itself. It would follow from this definition that 1 is a prime number. However, mathematicians do not consider 1 to be prime.

*

Answer for Part A. 2, 11, 3, 17, and 61 are prime.

*

Answers for Part B.

1. Since the number has itself and 1 as factors, if it has no other factors then it has at most two factors. Since it is not 1, it has exactly two factors. So, it is prime.

2. [We give two proofs. In each, 'G' names the set of positive-integers-greater-than-1.]

   Suppose that n is a positive integer other than 1 which has no factor with respect to G. It follows that if there are positive integers x and y such that $n = xy$ and $x \in G$, then $y \notin G$. Since y is a positive integer, it follows that $y = 1$. So, $x = n$. Hence, the only factor of n, other than 1, with respect to the set of positive integers, is n itself. With respect to the set of positive integers, n has no factors other than itself and 1. Since n is a positive integer other than 1, it follows from the theorem in Exercise 1 that n is a prime number. Therefore, if n is a positive integer other than 1 which has no factor with respect to G then n is a prime number. [End of first proof.]

   Suppose that n is a positive integer other than 1 which has no factor with respect to G. Now, if there are positive integers x and y such that $n = xy$ and $x < n$, then $y > 1$. [For each x and y, if $0 < x < xy$ then $1 < y$. For, suppose that $x > 0$ and $x < xy$. Then, by division, $1 < y$.] Hence, if, with respect to the set of positive integers, n has a factor in G which is smaller than n, then this number is a factor of n with respect to G. Since, by assumption, n has no factor with respect to G, it follows that, with respect to the set of positive integers, 1 is the only factor of n which is smaller than n. But, each factor of n, with respect to the set of positive integers, is less than or equal to n. [For each x and y, if $x \geq 1$ and $y \geq 1$ then $xy \geq x$. For, if $x \geq 1$ then $x > 0$, and if, besides, $y \geq 1$, then (by multiplication) $xy \geq x$.] So, with respect to the set of positive integers, n has no factors other than itself and 1. Since n is a positive integer other than 1, it follows from the theorem of Exercise 1 that n is a prime number. Therefore, if n is a positive integer other than 1 which has no factors with respect to G then n is a prime number.

[4-56]

[4-57]

the product by the prime and inserting in it the decimal name for the prime. Since there is just one prime factorization of the product it follows that, however we find it, it will contain the decimal name for the given prime. Now, one way of finding the prime factorization of the product is to "combine" [in the obvious manner] the prime factorizations of the two integers. So, the decimal name for the given prime must occur in at least one of these two prime factorizations. Hence, the given prime is a factor of at least one of the integers.

*

You may wish to raise a question concerning the number of primes. If one decides to make a list of the primes by examining each positive integer starting with 2, he finds that he comes upon primes less and less frequently. It is natural to wonder if you would run out of primes. This is equivalent to asking:

Is there a largest prime?

or to asking:

Is there an infinite number of primes?

Euclid answered this question, and the proof he gave is very much like the one we shall give.

We shall first prove the lemma that, for each finite set of primes, there is a prime which does not belong to the set. Suppose S is a finite set of primes, $\{p_1, p_2, p_3, \ldots, p_m\}$. The number

$$p_1 p_2 p_3 \cdots p_m + 1$$

is a positive integer different from 1, and so it is either a prime or a product of primes. Now, none of the primes $p_1, p_2, p_3, \ldots, p_m$ is a factor of $p_1 p_2 p_3 \ldots p_m + 1$. So, if this number is a product of primes, there are primes which do not belong to S. On the other hand, if this number is itself a prime, it does not belong to S. So, if S is a finite set of primes, there is at least one prime which does not belong to it.

Now, let's prove that there is an infinite number of primes. Suppose the number of primes is finite, and suppose S is the set of all primes. By the lemma just proved, it follows that there is a prime which does not belong to S. [Contradiction.]

TC[4-57, 58]f

[4-57]

$m > 2$, $p_2 p_3 \ldots p_m$ is a product of primes and, since it has q_1 as a factor, there are primes $t_1, t_2, \ldots, t_k$ such that

$$p_2 p_3 \ldots p_m = q_1 t_1 t_2 \ldots t_k.$$

But, since $q_1 \neq p_i$ for $2 \leq i \leq m$, it follows that $p_2 p_3 \ldots p_m$ has two prime factorizations. This is impossible, since $p_2 p_3 \ldots p_m < N$.

This completes the proof, subject to verification of our conjecture that, for each j, $p_j \neq q_1$. Suppose $p_1 = q_1$. Then,

$$\frac{N}{p_1} = \frac{N}{q_1} = p_2 p_3 \ldots p_n = q_2 q_3 \ldots q_m.$$

Since $\frac{N}{p_1} < N$, $\frac{N}{p_1}$ has just one prime factorization.

Hence, $m = n$ and $p_2 = q_2$, $p_3 = q_3$, etc. But, since $p_1 = q_1$, this contradicts the assumption that we have two prime factorizations for N. Suppose, for some $j > 1$, $p_j = q_1$. Then,

$$\frac{N}{p_j} = \frac{N}{q_1} = p_1 p_2 \ldots p_{j-1} p_{j+1} \ldots p_n = q_2 q_3 \ldots q_m.$$

Since $\frac{N}{p_j} < N$, $\frac{N}{p_j}$ has just one prime factorization.

Hence, $m = n$ and $p_1 = q_2$, $p_2 = q_3$, $\ldots p_{j-1} = q_j$, $p_{j+1} = q_{j+1}$, $\ldots$, $p_n = q_m$. But, since $p_1 = q_j$, this contradicts the assumption that we have two prime factorizations for N.

*

From this theorem one can derive many others. Here is one of the more useful ones.

> If a prime number is a factor of the product of two positive integers then it is a factor of one of the integers.

For, if a prime is a factor of the product, one can find a prime factorization of the product by finding a prime factorization of the quotient of

[4-57]

*

It is asserted near the end of Exercise 8 on page 4-58 that each positive integer other than 1 has just one prime factorization. We give below a proof of this assertion.

First, we need a definition of 'prime factorization'. A prime factorization is a numeral consisting of [decimal] numerals for prime numbers connected by multiplication signs, and such that of two of these numerals the one to the right of the other does not name a smaller number. For example, a prime factorization of 60 is '$2 \cdot 2 \cdot 3 \cdot 5$'. But, '$2 \cdot 3 \cdot 2 \cdot 5$' is not.

Next, we prove the lemma:

> Each positive integer other than 1 is either a prime or a product of primes.

If this is not the case then there are positive integers other than 1 which are neither prime nor products of primes. If there are such positive integers then there is a smallest one, N. Since N is not a prime, there is a positive integer n such that $1 < n < N$ and n is a factor of N. Consequently, $1 < N \div n < N$. So, since N is the smallest positive integer other than 1 which is neither a prime nor a product of primes, it must be the case that each of the numbers n and $N \div n$ is either a prime or a product of primes. But,

$$N = n \cdot (N \div n).$$

So, N is a product of primes. [Contradiction.]

Now, we are ready to prove the theorem:

> No positive integer has two prime factorizations.

If this is not the case then there is a smallest positive integer N which has two prime factorizations. [It is clear that each prime number has just one prime factorization.] So, there are m prime numbers $p_1, p_2, \ldots, p_m$, and n prime numbers $q_1, q_2, \ldots, q_n$ [$m > 1$, $n > 1$] such that

$$N = p_1 p_2 \cdots p_m = q_1 q_2 \cdots q_n,$$

[4-57]

☆6. The numbers are 6, 8, 10, 14, 15, 21, 22, 26, 27. [They are:

(a) the product of two primes,

or:

(b) the cubes of primes.]

*

Answers for Part D [on pages 4-57 and 4-58].

1. 1, 2, 3, 4, 6, 8, 12, and 24.

[In Exercises 2 through 7, you should expect some students to be able to characterize [as shown below] the classes of numbers which can serve as answers.]

2. A product of two primes. [For example: 6.]

3. The square of a prime.

4. The cube of a prime ["a prime × itself × itself"].

5. The fourth power of a prime ["a prime × itself × itself × itself", or "the square of the square of a prime"]. [For example: 16. The factors of 16 are 1, 2, 2×2, $2 \times 2 \times 2$, and $2 \times 2 \times 2 \times 2$.]

6. The sixth power of a prime.

7. 6; any product of powers of 2 and of 3. [For example: 72. $72 = (2 \times 2 \times 2) \times (3 \times 3)$, and its only prime factors are 2 and 3.]

8. (a) $2 \cdot 7 \cdot 7$ (b) $2 \cdot 2 \cdot 3 \cdot 3 \cdot 3$ (c) $7 \cdot 11 \cdot 11$

(d) $2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$ (e) $2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \cdot 5$ (f) $17 \cdot 53$

[A systematic way consists in dividing by 2, dividing the quotient by 2, dividing that quotient by 2, . . . , dividing that quotient by 3, dividing that quotient by 3, . . . , by 5, . . . , by 7, . . . , by 11, ...]

9. (a) 1, 2, 7, 14, 49, 98

(b) 1, 2, 3, 4, 6, 9, 12, 18, 27, 36, 54, 108

(c) 1, 7, 11, 77, 121, 847

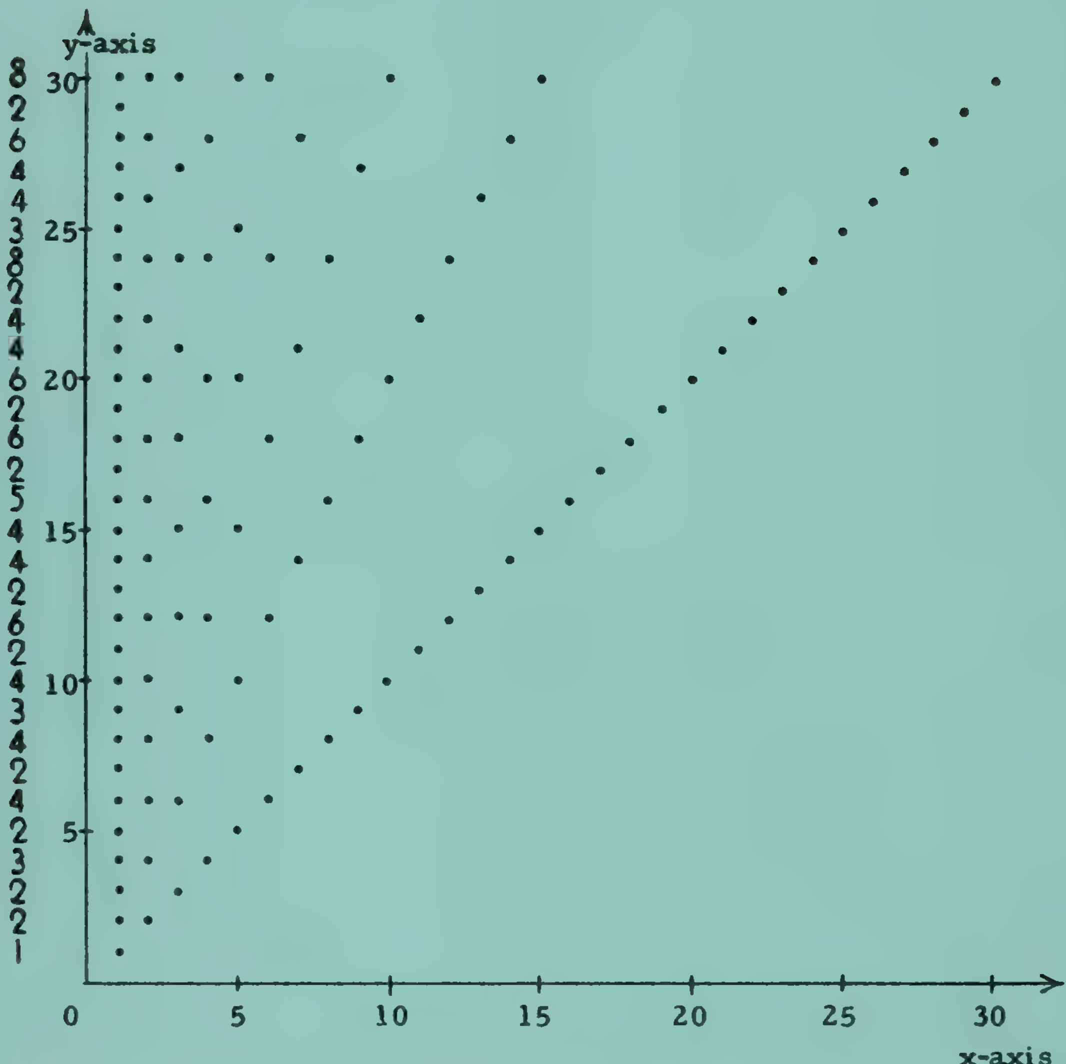
(d) 1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 36, 48, 72, 144

(e) 1, 2, 3, 4, 5, 6, 9, 10, 12, 15, 18, 20, 25, 30, 36, 45, 50, 60, 75, 90, 100, 150, 180, 225, 300, 450, 900

(f) 1, 17, 53, 901

[4-57]

Answers for Part C.



Exercises 1 and 2 are answered by the above chart.

3. 1

4. 2, 3, 5, 7, 11, 13, 17, 19, 23, 29. [They are the prime numbers less than 30.]

5. The numbers are 4, 9, and 25. [They are the squares of the primes each of whose squares is less than 30.]

C. 1. On a picture of "Quadrant I of the number plane lattice", draw a picture of {(x, y): x is a factor of y with respect to the set of positive integers}. [Choose your scale so that your picture includes the graph of (30, 30).]

2. On each horizontal line indicate, by writing a numeral to the left of the vertical axis, the number of factors of the corresponding number.

3. How many numbers have just one factor with respect to the set of positive integers?

4. Which numbers have just two factors with respect to the set of positive integers?

5. Describe the numbers which have just three factors with respect to the set of positive integers.

☆ 6. Repeat Exercise 5 for four factors, instead of three.

D. A positive integer which is neither a prime number nor 1 is a composite number. Each composite number has at least three factors with respect to the set of positive integers [Explain]. Give five examples of composite numbers.

*

Let's agree that from now on when we talk about factors of numbers we mean factors with respect to the set of positive integers (unless we say otherwise).

*

1. All of the factors of the composite number 12 are 1, 2, 3, 4, 6, and 12. List all numbers which are factors of the composite number 24.

2. Find a composite number which has just four factors exactly two of which are prime factors. [A prime factor is a factor which is a prime number. For example, 2 is a prime factor of 16.]

3. Find a composite number which has exactly three factors only one of which is a prime factor.

(continued on next page)

4. Find a composite number which has exactly four factors only one of which is a prime factor.

5. Find a composite number which has exactly five factors only one of which is a prime factor.

6. Find a composite number which has exactly seven factors only one of which is a prime factor.

7. Find a composite number which has 2 and 3 as its only prime factors. Find four more such composite numbers.

8. The factors of 72 are 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, and 72. Of these, the only prime factors are 2 and 3. We can use these prime factors to factor '72'.

$$72 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$$

We shall call the expression '$2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$' <u>the prime factorization of the number 72</u>. So, the prime factorization of a positive integer is a particular kind of numeral for the integer. Here are other examples of prime factorizations.

$18 = 2 \cdot 3 \cdot 3$	$40 = 2 \cdot 2 \cdot 2 \cdot 5$	$35 = 5 \cdot 7$
$130 = 2 \cdot 5 \cdot 13$	$1400 = 2 \cdot 2 \cdot 2 \cdot 5 \cdot 5 \cdot 7$	$29 = 29$

It can be proved that each positive integer other than 1 has just one prime factorization.

Give the prime factorization for each of the listed numbers. [Can you find a systematic way of solving exercises like these?]

(a) 98	(b) 108	(c) 847
(d) 144	(e) 900	(f) 901

9. Use the prime factorizations you found in Exercise 8 in making a list of all the factors of each of the given numbers. [Can you find a systematic way of solving exercises like these?]

(a) 98	(b) 108	(c) 847
(d) 144	(e) 900	(f) 901

[4-59]

s 'xx'

nt

Similar considerations arise in connection with Exercise 7, if students give '$(ab^2)^2a$' for an answer, and with Exercise 6, if students give '$n^6 - (r^2s)^2$' for an answer. Answers for Exercises 5, 6, and 7 such as 'x^4y^4', '$n^6 - r^4s^2$', and 'a^3b^4', require as justification the cpm as well as the apm. Needless to say, students should be allowed to make tacit use of these principles.

*

Answers for Part A.

1. 6^3a^4	2. x^4y^4	3. $36^2 + x^3$
4. $y^4 - a^4$	5. $(xy)^4$	6. $(n^3)^2 - (r^2s)^2$
7. $(ab^2)^2a$	8. $(a + b^2)^2$	9. $\dfrac{2^3x^2 - y^5}{3^2x^3 + 6^4y}$

*

Answers for Part B.

1. $2^2 \cdot 3^3 \cdot 5^2$	2. $2^6 \cdot 3^4$	3. $2^3 \cdot 3^2 \cdot 5^5$
4. $2^2 \cdot 7^2 \cdot 13$	5. $2^2 \cdot 3^2 \cdot 5^2 \cdot 11 \cdot 13$	6. $2^6 \cdot 5^6$

s 'xx'

nt

There are a few spots here which require a bit of care. Brief mention in class is all that is needed.

Strictly speaking, going from

$$`2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 4 \cdot 4 \cdot 4 \cdot 4' \text{ to } `2^3 \cdot 3^2 \cdot 4^4'$$

requires more than just abbreviations. Because of the convention concerning the omission of grouping symbols in products, the first expression can be abbreviated to:

$$2^3 \cdot 3 \cdot 3 \cdot 4 \cdot 4 \cdot 4 \cdot 4.$$

Further abbreviation requires the preliminary and repeated use of the associative principle for multiplication. Thus,

$$\begin{array}{ll}
 & 2^3 \cdot 3 \cdot 3 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \\
 & \qquad\qquad\qquad\qquad \text{[convention]} \\
= & [(\{[(2^3 \cdot 3) \cdot 3] \cdot 4\} \cdot 4) \cdot 4] \cdot 4 \\
 & \qquad\qquad\qquad\qquad \text{apm} \\
= & [(\{[2^3 \cdot (3 \cdot 3)] \cdot 4\} \cdot 4) \cdot 4] \cdot 4 \\
 & \qquad\qquad\qquad\qquad \text{[abbreviation]} \\
= & [(\{[2^3 \cdot 3^2] \cdot 4\} \cdot 4) \cdot 4] \cdot 4 \\
 & \qquad\qquad\qquad\qquad \text{apm} \\
= & [([2^3 \cdot 3^2] \cdot \{4 \cdot 4\}) \cdot 4] \cdot 4 \\
 & \qquad\qquad\qquad\qquad \text{apm} \\
= & [[2^3 \cdot 3^2] \cdot (\{4 \cdot 4\} \cdot 4)] \cdot 4 \\
 & \qquad\qquad\qquad\qquad \text{apm} \\
= & [2^3 \cdot 3^2] \cdot [(\{4 \cdot 4\} \cdot 4) \cdot 4] \\
 & \qquad\qquad\qquad\qquad \text{[convention]} \\
= & [2^3 \cdot 3^2] \cdot [4 \cdot 4 \cdot 4 \cdot 4] \\
 & \qquad\qquad\qquad\qquad \text{[abbreviation]} \\
= & [2^3 \cdot 3^2] \cdot 4^4 \\
 & \qquad\qquad\qquad\qquad \text{[convention]} \\
= & 2^3 \cdot 3^2 \cdot 4^4 \; .
\end{array}$$

You may want to go through this <u>once</u> for your class.

4.05 Exponents.--In Unit 3 we abbreviated expressions such as 'xx' to 'x^2', and '3 · 3' to '3^2'. The raised numeral is called an exponent symbol [or: an exponent]. We can use numerals for other positive integers as exponent symbols in order to simplify more complicated expressions. For example, we can abbreviate

'5 · 5 · 5' to '5^3', 'zzzz' to 'z^4',

'$\frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3}$' to '$\left(\frac{2}{3}\right)^4$' '(xx)(xx)' to '$(x^2)^2$',

'2 · 2 · 2 (3 · 3)(4 · 4 · 4 · 4)' to '$2^3 \cdot 3^2 \cdot 4^4$',

'(2x + y)(2x + y)(2x + y)' to '$(2x + y)^3$',

'aaaaaaaaaaaa' to 'a^{12}',

and '(xxy)(xxy)y' to '$(x^2y)^2y$'.

EXERCISES

A. Use exponents to abbreviate each expression.

1. $6 \cdot 6 \cdot 6(aaaa)$
2. xxxx(yyyy)
3. $36 \cdot 36 + xxx$
4. yyyy - aaaa
5. (xy)(xy)(xy)(xy)
6. (nnn)(nnn) - (rrs)(rrs)
7. (abb)(abb)a
8. (a + bb)(a + bb)
9. $\dfrac{2 \cdot 2 \cdot 2(xx) - yyyyy}{3 \cdot 3(xxx) + 6 \cdot 6 \cdot 6 \cdot 6y}$

[Supplementary exercises are in Part J on page 4-125.]

B. Use exponent notation to abbreviate prime factorizations for the listed numbers.

Sample. 5760

Solution. $5760 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5$
$= 2^7 \cdot 3^2 \cdot 5.$

1. 2700
2. $4^3 \cdot 9^2$
3. $10^3 \cdot 15^2$
4. 2548
5. $1287 \cdot 10^2$
6. 1000000

* * *

Using the commutative and associative principles for multiplication, you can see that

$$(5 \cdot 4)(5 \cdot 4) = 5^2 \cdot 4^2, \quad 8^3 \cdot 8^2 = 8^5, \quad (x^3 y^2)^2 = x^6 y^4,$$

and that '$z^3 \cdot z^2 \cdot z$' and 'z^6' are equivalent.

Also, $\frac{4^7}{4^3} = 4^4$, and '$\frac{x^3}{x^5}$' and '$\frac{1}{x^2}$' are equivalent, $[x \neq 0]$.

* * *

C. Examine the following expressions, and sort them into sets of equivalent expressions such that all the expressions (and only those) equivalent to a given expression are in the same set with it. [The domains of the pronumerals are such that no denominator can be converted into a name for 0.]

$y \cdot y^5$	$x^3 y^4$	$\left(\frac{8}{2}\right)^2$	the square of 6	$y(y^2)^2$	$2 \cdot 2^2$		
$2 \cdot 3^2$	$x^2 \cdot x$	$y^2 y^2 y^2$	$xy^2 x^2 y^2$	$xx^2 x$	$4 + 2$	y^5	
$y^5 \cdot y$	$2^5 \div 2$	x^3	y^6	$x^2 \cdot x^2$	$x^3 \cdot x$	4^2	xy^2
$2 \cdot 2 \cdot 3^2$	$y^2 \cdot y^3 \cdot y$	$(y^2)^2$	$\frac{8^2}{2^2}$	$y^2 \cdot y^2 \cdot y$	$(xy)^2$	4	
$(x^2 y)^2 y$	$(x^2)^2 y^3$	$6 \cdot 6$	$2^2 \cdot 3^2$	the square of y^2	6^2		
$2(2)^2$	$\frac{4^2}{2}$	the square of $(2 \cdot 3)$	$\frac{x^4 y^4}{x^2 y^2}$	$\frac{2^8}{2^4}$	$x^2 y^2 x^2 y$		
$(xy)^3 y$	$x^4 y^3$	$xy^2 \cdot xy^2 \cdot x$	$(xy)^3 x$	$y^4 y^2$	$2 \cdot 2 \cdot 2 \cdot 2$		
$\frac{x^2 y^3}{xy}$	$x^4 y^3 \cdot y^3$	$3 \cdot 2^2 \cdot 3$	$9 \cdot 4$	xyy	$(y^2)^3$		
the square of the square of x	$\frac{x^4 y^4}{x^3 y^2}$	$x(x^2)(y^4)$	$(x^2)^2$				
$2 \cdot 2 \cdot 2$	xxx	36	$2^2 \cdot 2^2$	$\frac{8^2}{2^3}$	$\frac{x^6 y^6}{x^4 y^4}$	$x(xy^2)^2$	
8	16	$\frac{12^2}{2^2}$	$y(xy)$	$2(2 \cdot 3^2)$	$xx \cdot yy$	$(xy^2)x$	

Answers for Part C.

$y \cdot y^5$ $\quad y^2y^2y^2$ $\quad y^5 \cdot y$ $\quad y^6$ $\quad y^2 \cdot y^3 \cdot y$ $\quad y^4 \cdot y^2$ $\quad (y^2)^3$

---

x^3y^4 $\quad xy^2x^2y^2$ $\quad (xy)^3y$ $\quad xy^2 \cdot xy^2 \cdot x$ $\quad x(x^2)(y^4)$ $\quad x(xy^2)^2$

$\left(\frac{8}{2}\right)^2$ $\quad 2^5 \div 2$ $\quad 4^2$ $\quad \frac{8^2}{2^2}$ $\quad \frac{2^8}{2^4}$ $\quad 2^2 \cdot 2^2$ $\quad 2 \cdot 2 \cdot 2 \cdot 2$ $\quad 16$

---

square of 6 $\quad 2 \cdot 2 \cdot 3^2$ $\quad 6 \cdot 6$ $\quad 2^2 \cdot 3^2$ $\quad 6^2$ $\quad 3 \cdot 2^2 \cdot 3$ $\quad 36$

$\frac{12^2}{2^2}$ $\quad$ square of $(2 \cdot 3)$ $\quad 9 \cdot 4$ $\quad 2(2 \cdot 3^2)$

$y(y^2)^2$ $\quad y^5$ $\quad y^2 \cdot y^2 \cdot y$

$2 \cdot 2^2$ $\quad 2(2)^2$ $\quad \frac{4^2}{2}$ $\quad 2 \cdot 2 \cdot 2$ $\quad \frac{8^2}{2^3}$ $\quad 8$

$x^2 \cdot x$ $\quad x^3$ $\quad xxx$

---

xx^2x $\quad x^2 \cdot x^2$ $\quad x^3 \cdot x$ $\quad (x^2)^2$ $\quad$ square of the square of x

---

xy^2 $\quad \frac{x^2y^3}{xy}$ $\quad xyy$ $\quad \frac{x^4y^4}{x^3y^2}$ $\quad y(xy)$

$(xy)^2$ $\quad \frac{x^4y^4}{x^2y^2}$ $\quad \frac{x^6y^6}{x^4y^4}$ $\quad xx \cdot yy$ $\quad (xy^2)x$

$(x^2y)^2y$ $\quad (x^2)^2y^3$ $\quad x^2y^2x^2y$ $\quad x^4y^3$ $\quad (xy)^3x$

$2 \cdot 3^2$	$4 + 2$	$(y^2)^2$ $\quad$ square of y^2	4	$x^4y^3 \cdot y^3$

[4-61]

The major purpose of Part E is to make students aware of certain "short cuts" which are erroneous. The discussion of the answers given on the preceding page deals only with pointing out the errors. It is also important that students know how to correct the errors. So, for example, in Exercise 2 you might ask about the change which could be made in the first expression to make it equivalent to the second [Answer: change it to '2^2a^2', or: change it to '$4a^2$'.].

In Exercise 3, we could change 'xy^2' to 'x^2y^2'.

In Exercise 4, ask if '$x^4 \cdot x^5$' is equivalent to '$x^5 \cdot x^4$'. Ask if '$x^4 + x^5$' is equivalent to '$x^5 + x^4$'. Ask if '$x^4 \cdot y^5$' and '$x^5 \cdot y^4$' are equivalent; if '$x^4 + y^5$' and '$x^5 + y^4$' are equivalent.

In Exercise 5, ask if the operation squaring is distributive with respect to addition; if squaring is distributive with respect to multiplication. With respect to division. With respect to subtraction.

In Exercise 6, change 'a^5b^7' to 'a^6b^{10}'.

In Exercise 10, change 'a + b' to '$\frac{a}{b} + \frac{b}{a}$'. Or, change '$\frac{a^2 + b^2}{ab}$' to '$\frac{a^2b + ab^2}{ab}$'.

In Exercise 11, change '$\frac{a^2 + b^2}{a + b}$' to '$\frac{a^2 + 2ab + b^2}{a + b}$'. Or, change 'a + b' to '$\frac{a^2}{a + b} + \frac{b^2}{a + b}$'.

In Exercise 13, ask:

> In claiming that '$\sqrt{x^4}$' is equivalent to 'x^2', haven't we violated the generalization that $\forall_x \sqrt{x^2} = |x|$?

We haven't, of course, because, for each x, x^2 is nonnegative. This point is brought out nicely in the contrast between Exercises 13 and 14, and also between Exercises 13 and 15. In Exercise 15, students might jump to the conclusion that if '6ab' were changed to '$6ab^2$', the resulting expressions would be equivalent. Such a conclusion is incorrect. If we had '$6|a|b^2$', the two expressions would be equivalent.

In Exercise 16, we should have '$1/x^4$' instead of '−4'. This exercise might provoke a student to ask if 'x^{-4}' means something. Expressions such as '3^{-2}' and '8^{-5}' must be <u>defined</u>, the definitions being chosen in such a way that certain manipulation rules hold for the newly-defined symbols. Our unit dealing with exponents treats this question at great length, and there is a very brief treatment of it on page 4-67.

[4-61]

3. nonequivalent [Using '2' for 'x' and '1' for 'y', the corresponding value of '$(xy)^2$' is $(2 \cdot 1)^2$, or 4. The corresponding value of 'xy^2' is $2 \cdot 1^2$, or 2.]
4. equivalent
5. nonequivalent [Using '1' for 'x', the corresponding value of '$(x + 2)^2$' is $(1 + 2)^2$, or 9. The corresponding value of '$x^2 + 4$' is $1^2 + 4$, or 5.]
6. nonequivalent [Using '1' for 'a' and '2' for 'b', the corresponding value of '$(a^3b^5)^2$' is $(1^3 \cdot 2^5)^2$, or 1024. The corresponding value of 'a^5b^7' is $1^5 \cdot 2^7$, or 128.]
7. equivalent
8. equivalent
9. equivalent
10. nonequivalent [Using '1' for 'a' and '2' for 'b', the corresponding value of '$\frac{a^2 + b^2}{ab}$' is $\frac{1^2 + 2^2}{1 \cdot 2}$, or $\frac{5}{2}$. The corresponding value of '$a + b$' is $1 + 2$, or 3.]
11. nonequivalent [Using '1' for 'a' and '2' for 'b', the corresponding value of '$\frac{a^2 + b^2}{a + b}$' is $\frac{1^2 + 2^2}{1 + 2}$, or $\frac{5}{3}$. The corresponding value of '$a + b$' is $1 + 2$, or 3.]
12. equivalent
13. equivalent
14. nonequivalent [Using '-2' for 'y', the corresponding value of '$\sqrt{y^6}$' is $\sqrt{(-2)^6}$, or 8. The corresponding value of 'y^3' is $(-2)^3$, or -8.]
15. nonequivalent [Using '-2' for 'a' and '3' for 'b', the corresponding value of '$\sqrt{(-3ab^2)(-12ab^2)}$' is $\sqrt{(-3 \cdot -2 \cdot 3^2)(-12 \cdot -2 \cdot 3^2)}$, that is, 108. The corresponding value of '6ab' is $6 \cdot -2 \cdot 3$, or -36.]
16. nonequivalent [Using '1' for 'x', the corresponding value of '$x^9 \div x^{13}$' is $1^9 \div 1^{13}$, or 1. The corresponding value of '-4' is -4.]

[4-61]

34. c^{20}

35. x^{20}

36. $(-2)^3x^6$ [or: $-8x^6$]

37. $(-3)^4y^{12}$ [or: $81y^{12}$]

38. $(-2)^2x^6$ [or: $4x^6$]

39. $(-3)^3y^{12}$ [or: $-27y^{12}$]

40. $9^2x^6y^8$ [or: $81x^6y^8$]

41. $7^3a^{12}b^6$ [or: $343a^{12}b^6$]

42. $2^2a^4b^5$ [or: $4a^4b^5$]

43. $3^2c^5d^4$ [or: $9c^5d^4$]

44. $\left(-\frac{1}{2}\right)^3a^{15}$ [or: $-\frac{1}{8}a^{15}$]

45. $\left(-\frac{1}{3}\right)^2b^8$ [or: $\frac{1}{9}b^8$]

46. $\frac{y^5}{3^3}$ [or: $\frac{y^5}{27}$]

47. $-\frac{a^5}{2^7}$ [or: $-\frac{a^5}{128}$]

48. $-\frac{2}{5^2}x^5$ [or: $-\frac{2}{25}x^5$]

49. b^6

50. $3^3x^6y^9z^{12}$ [or: $27x^6y^9z^{12}$]

51. $2^3a^{12}b^6c^6$ [or: $8a^{12}b^6c^6$]

52. $(-2)^5r^{20}s^{25}t^{30}$ [or: $-32r^{20}s^{25}t^{30}$]

53. $3^4r^{20}s^{24}t^{28}$ [or: $81r^{20}s^{24}t^{28}$]

54. x^{12}

55. a^{40}

56. $2^2 \cdot 3^2n^{10}s^{10}$ [or: $36n^{10}s^{10}$]

57. $2^2 \cdot 3^2c^{10}d^{10}$ [or: $36c^{10}d^{10}$]

*

The directions for Part E ask the student to give a counter-example for those cases in which he thinks a pair of expressions are nonequivalent. This is a simple matter for such an exercise as 2 or 5, in which the given expressions contain only one pronumeral. But in Exercise 3 [and others whose expressions contain two pronumerals], the job of citing a counter-example is a bit more involved. This complication has been discussed on TC[2-30]a and b.

*

Answers for Part E [on page 4-64]. [Be sure to see the additional discussion which follows these answers.]

1. equivalent

2. nonequivalent [Using '1' for 'a', the corresponding value of '$2a^2$' is $2 \cdot 1^2$, or 2. The corresponding value of '$(2a)^2$' is $(2 \cdot 1)^2$, or 4.]

[4-61]

Therefore, before the theorems as stated above [which include 1 in the domain of 'm' and of 'n'] can make sense, you will have to define exponent expressions in which '1' occurs as an exponent. This definition has been postponed in the text until page 4-67, but you will have to introduce it now if you state these theorems.

Some students are apt to offer as simplifications for Exercise 21 either 'c^{11}' or 'c^{20}'. As in the case of all errors of simplification, you can handle this by asking if the student believes the generalization:

$$\forall_c \quad c^9 + c^2 = c^{11}.$$

If so, he should believe each instance, and be able to give a testing pattern. A counter-example is usually enough to shake him, but discovering the error in his alleged testing pattern will be even more helpful.

*

Answers for Part D [on pages 4-61, 4-62, and 4-63].

1. a^{15} 2. 3^9 3. 2^{11} 4. x^{10} 5. $6n^5$

6. $-14r^7$ 7. $-9a^3b^3$ 8. $-20c^3d^5$ 9. $.2nr^3s$ 10. $.18c^2d^3e$

11. x^3, $[x \neq 0]$ 12. 4^2 [or: 16] 13. -7^3 [or: -343]

14. $-r^5$, $[r \neq 0]$ 15. $\frac{x}{2y^2}$, $[xy \neq 0]$ 16. $3m$, $[hm \neq 0]$

17. $-\frac{7a}{bc}$, $[abc \neq 0]$ 18. $\frac{-8c^3}{e}$, $[cde \neq 0]$ 19. 1, $[xyz \neq 0]$

20. $\frac{-b^2}{2}$, $[abc \neq 0]$ 21. $c^9 + c^2$ 22. $x^9 - x^8$

23. $8x^5y$ 24. $15y^4z$ 25. $2a^3b^3$

26. $3c^3d^3$ 27. 14^4 [or: 38, 416] 28. 32^2 [or: 1024]

29. $\frac{-5n^2s}{t^3}$, $[nst \neq 0]$ 30. $-\frac{5j^2k}{h}$, $[hjk \neq 0]$

31. $\frac{6ac^3xy^2}{5b^2z^3}$, $[abcxyz \neq 0]$ 32. $\frac{3ap}{2bmq^2}$, $[abcmpq \neq 0]$

33. $\frac{4(a+b)(x+y)^2(r+s)(u+v)^2}{3}$, $[(r+s)(u+v)(x+y)(a+b) \neq 0]$

[4-61]

There is some danger in giving the usual colloquial statements of the rules for manipulating exponent expressions. For example, a student who parrots "to multiply you add the exponents" is likely to simplify '$x^3 y^2$' to '$(xy)^5$' or '$(x^3)^2$' to 'x^5'. If students are <u>eager</u> to state the generalizations [short cuts] of which they have become aware at the nonverbal level while working through the exercises of Part D, be sure to arrive at precise statements such as:

For each real number x, for each positive integer m, for each positive integer n,

$$x^m \cdot x^n = x^{m+n},$$

For each real number x, for each positive integer m, for each positive integer n,

$$(x^m)^n = x^{mn}.$$

For each real number x, for each real number y, for each positive integer m,

$$(xy)^m = x^m y^m.$$

For each real number $x \neq 0$, for each positive integer m, for each positive integer n,

(1) if $m > n$, $\frac{x^m}{x^n} = x^{m-n}$,

(2) if $m < n$, $\frac{x^m}{x^n} = \frac{1}{x^{n-m}}$, and

(3) if $m = n$, $\frac{x^m}{x^n} = 1$.

These theorems are stated and proved in a later unit on exponents. The proofs require mathematical induction.

Note carefully that some consequences of these theorems are sentences such as:

$2^1 \cdot 2^5 = 2^6$ and: $(x^1 y^2)^3 = x^3 y^6$.

D. Simplify. [Look for short cuts.]

Sample 1. $(4a^6)6a^4)$

Solution. $(4a^6)(6a^4) = (4 \cdot 6)(a^6 \cdot a^4)$

$= 24[(aaaaaa)(aaaa)]$

$= 24(aaaaaaaaaa)$

$= 24a^{10}$

Sample 2. $\dfrac{2x^{12}y^3}{5x^{15}y}$

Solution. $\dfrac{2x^{12}y^3}{5x^{15}y} = \dfrac{2(xxxxxxxxxxxx)(yyy)}{5(xxxxxxxxxxxxxxx)y}$

$= \dfrac{2(yy)(xxxxxxxxxxxxy)}{5(xxx)(xxxxxxxxxxxxy)}$

$= \dfrac{2(yy)}{5(xxx)}$

$= \dfrac{2y^2}{5x^3}$, $[x \neq 0,\ y \neq 0]$.

1. $a^5 \cdot a^{10}$
2. $3^4 \cdot 3^5$
3. $2^3 \cdot 2^2 \cdot 2 \cdot 2^5$
4. $x^2 \cdot x^3 \cdot x \cdot x^4$
5. $(2n^2)(3n^3)$
6. $(-7r^3)(2r^4)$
7. $(3a^2b)(-3ab^2)$
8. $(5cd^2)(-4c^2d^3)$
9. $(.4nr^2)(.5rs)$
10. $(.3c^2d)(.6d^2e)$
11. $\dfrac{x^5}{x^2}$
12. $\dfrac{4^4}{4^2}$
13. $\dfrac{-7^5}{7^2}$
14. $\dfrac{r^8}{-r^3}$
15. $\dfrac{3x^2y}{6xy^3}$
16. $\dfrac{6hm^2}{2hm}$
17. $\dfrac{21a^2b^3c}{-3ab^4c^2}$
18. $\dfrac{-32cd^2c^4}{4c^2d^2e}$
19. $\dfrac{-7x^2y^5z}{-7x^2y^5z}$
20. $\dfrac{-12a^3b^3c}{24a^3bc}$
21. $(c^6)(c^3) + c^2$
22. $(x^4)(x^5) - x^8$

(continued on next page)

23. $(-2x^2)(-4x^3y)$ 24. $(-3y^3)(-5yz)$ 25. $(a^2b)(2ab^2)$

26. $(cd^2)(3c^2d)$ 27. $\frac{14^6}{14^2}$ 28. $\frac{32^5}{32^3}$

29. $\frac{-15n^3s^2t}{3nst^4}$ 30. $\frac{20h^3j^4k^5}{-4h^4j^2k^4}$

31. $\frac{5x^2y^3z^2}{4ab^3c^2} \times \frac{24a^2bc^5}{25xyz^5}$ 32. $\frac{-2mp^2q}{3a^2b^3c^5} \times \frac{-9a^3b^2c^5}{4m^2pq^3}$

33. $\frac{3(a+b)^5(x+y)^7}{7(r+s)\ (u+v)^5} \times \frac{28(r+s)^2(u+v)^7}{9(x+y)^5(a+b)^4}$

<u>Sample 3.</u> $(x^2)^3$

<u>Solution.</u>
$$(x^2)^3 = x^2 \cdot x^2 \cdot x^2$$
$$= (xx)(xx)(xx)$$
$$= x^6 .$$

<u>Sample 4.</u> $(2x)^5$

<u>Solution.</u>
$$(2x)^5 = (2x)(2x)(2x)(2x)(2x)$$
$$= (2 \cdot 2 \cdot 2 \cdot 2 \cdot 2)(xxxxx)$$
$$= 2^5 x^5 \ [\text{or: } 32x^5].$$

<u>Sample 5.</u> $\left(-\frac{x}{2}\right)^3 (2x)^4$

<u>Solution.</u>
$$\left(-\frac{x}{2}\right)^3 (2x)^4 = \left[\left(-\frac{1}{2}\right)^3 x^3\right][2^4 x^4]$$
$$= -\frac{x^3}{2^3} \cdot 2^4 x^4$$
$$= -2x^7 .$$

34. $(c^4)^5$ 35. $(x^4)^5$ 36. $(-2x^2)^3$

37. $(-3y^3)^4$ 38. $(-2x^3)^2$ 39. $(-3y^4)^3$

40. $(9x^3y^4)^2$

41. $(7a^4b^2)^3$

42. $(a^2b)(2ab^2)^2$

43. $(cd^2)(3c^2d)^2$

44. $\left(-\frac{1}{2}a^5\right)^3$

45. $\left(-\frac{1}{3}b^4\right)^2$

46. $\left(\frac{y}{3}\right)^3 y^2$

47. $\left(-\frac{a}{4}\right)^2\left(-\frac{a}{2}\right)^3$

48. $\left(\frac{1}{5}x\right)^2(-2x^3)$

49. $\left(-\frac{1}{4}b\right)^3(-4b)^3$

50. $(3x^2y^3z^4)^3$

51. $(2a^4b^2c^2)^3$

52. $(-2r^4s^5t^6)^5$

53. $(-3r^5s^6t^7)^4$

54. $(x^2)^3(x^3)^2$

55. $(a^4)^5(a^5)^4$

56. $(2n^2s^3)^2(3n^3s^2)^2$

57. $(-3c^3d^2)^2(2c^2d^3)^2$

[Supplementary exercises are in Part K on page 4-126 and in Part M on page 4-128.]

E. Each exercise contains two pronumeral expressions. For each pair of expressions, tell whether the expressions are equivalent or nonequivalent. If they are nonequivalent, give a counter-example.

1. $(x^2)^3$, x^6

2. $2a^2$, $(2a)^2$

3. $(xy)^2$, xy^2

4. $(x^4)^5$, $(x^5)^4$

5. $(x + 2)^2$, $x^2 + 4$

6. $(a^3b^5)^2$, a^5b^7

7. $(2z)^2$, $4z^2$

8. $y^3 \cdot y^2$, y^5

9. $\frac{a^2x^4}{(ax)^3}$, $\frac{x}{a}$, $[ax \neq 0]$

10. $\frac{a^2 + b^2}{ab}$, $a + b$, $[ab \neq 0]$

(continued on next page)

11. $\frac{a^2 + b^2}{a + b}$, $a + b$, $[a \neq -b]$

12. $\frac{a^2 - b^2}{a + b}$, $a - b$, $[a \neq -b]$

13. $\sqrt{x^4}$, x^2

14. $\sqrt{y^6}$, y^3

15. $\sqrt{(-3ab^2)(-12ab^2)}$, $6ab$

16. $x^9 \div x^{13}$, -4, $[x \neq 0]$

[Supplementary exercises are in Part L on page 4-127.]

F. The number 8 is called the third power of 2 because $8 = 2^3$. Complete each of the following sentences.

1. 16 is the _______ power of 2 because $16 = 2^{-}$.
2. _______ is the third power of 7 because _______ $= 7^3$.
3. 729 is the sixth power of _______ because $729 = \_\_\_^6$.
4. The tenth power of 2 is _______ because $\_\_\_^{-} = \_\_\_$.

G. A number which is either a prime number or a power of a prime number is called a prime power. Here is a list of some prime powers:

$$3^4, \quad 49, \quad 29^6, \quad 125, \quad 1024, \quad 4^3, \quad 53^2, \quad 11^{273}, \quad 41.$$

You have seen that you can name a positive integer by its prime factorization. You can also name a positive integer by its prime power factorization. For example:

$$72 = 2^3 \cdot 3^2, \qquad 98 = 2 \cdot 7^2, \qquad 624 = 2^4 \cdot 3 \cdot 13$$

$$900 = 2^2 \cdot 3^2 \cdot 5^2, \qquad 847 = 7 \cdot 11^2, \qquad 108 = 2^2 \cdot 3^3 .$$

1. Give the prime power factorization of each number.

   (a) 36 (b) 135 (c) 78

   (d) 216 (e) 981 (f) 6240

2. For each number listed in Exercise 1, list all of its factors, other than 1, by writing the prime power factorization for each factor. [Example: The factors, other than 1, of $2^2 \cdot 3^3$ are 3, 3^2, 3^3, 2, $2 \cdot 3$, $2 \cdot 3^2$, $2 \cdot 3^3$, 2^2, $2^2 \cdot 3$, $2^2 \cdot 3^2$, and $2^2 \cdot 3^3$.]

For
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Answers for Part F.

1. fourth, ^{4} 2. 343, 343

3. 3, 3 4. 1024, $2^{10} = 1024$

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Answers for Part G.

1. (a) $2^2 \cdot 3^2$ (b) $3^3 \cdot 5$ (c) $2 \cdot 3 \cdot 13$

(d) $2^3 \cdot 3^3$ (e) $3^2 \cdot 109$ (f) $2^5 \cdot 3 \cdot 5 \cdot 13$

2. (a) $2,\ 2^2,\ 3,\ 3^2,\ 2 \cdot 3,\ 2 \cdot 3^2,\ 2^2 \cdot 3,\ 2^2 \cdot 3^2$

(b) $3,\ 3^2,\ 3^3,\ 5,\ 3 \cdot 5,\ 3^2 \cdot 5,\ 3^3 \cdot 5$

(c) $2,\ 3,\ 13,\ 2 \cdot 3,\ 2 \cdot 13,\ 3 \cdot 13,\ 2 \cdot 3 \cdot 13$

(d) $2,\ 2^2,\ 2^3,\ 3,\ 3^2,\ 3^3,\ 2 \cdot 3,\ 2 \cdot 3^2,\ 2 \cdot 3^3,\ 2^2 \cdot 3,\ 2^2 \cdot 3^2,\ 2^2 \cdot 3^3,$
$2^3 \cdot 3,\ 2^3 \cdot 3^2,\ 2^3 \cdot 3^3$

(e) $3,\ 3^2,\ 109,\ 3 \cdot 109,\ 3^2 \cdot 109$

(f)

2	$2 \cdot 3$	$2 \cdot 5$	$2 \cdot 13$	$2 \cdot 3 \cdot 5$	$2 \cdot 3 \cdot 13$
2^2	$2^2 \cdot 3$	$2^2 \cdot 5$	$2^2 \cdot 13$	$2^2 \cdot 3 \cdot 5$	$2^2 \cdot 3 \cdot 13$
2^3	$2^3 \cdot 3$	$2^3 \cdot 5$	$2^3 \cdot 13$	$2^3 \cdot 3 \cdot 5$	$2^3 \cdot 3 \cdot 13$
2^4	$2^4 \cdot 3$	$2^4 \cdot 5$	$2^4 \cdot 13$	$2^4 \cdot 3 \cdot 5$	$2^4 \cdot 3 \cdot 13$
2^5	$2^5 \cdot 3$	$2^5 \cdot 5$	$2^5 \cdot 13$	$2^5 \cdot 3 \cdot 5$	$2^5 \cdot 3 \cdot 13$

$2 \cdot 5 \cdot 13$	$2 \cdot 3 \cdot 5 \cdot 13$	3	5	13
$2^2 \cdot 5 \cdot 13$	$2^2 \cdot 3 \cdot 5 \cdot 13$	$3 \cdot 5$	$5 \cdot 13$	
$2^3 \cdot 5 \cdot 13$	$2^3 \cdot 3 \cdot 5 \cdot 13$	$3 \cdot 13$		
$2^4 \cdot 5 \cdot 13$	$2^4 \cdot 3 \cdot 5 \cdot 13$	$3 \cdot 5 \cdot 13$		
$2^5 \cdot 5 \cdot 13$	$2^5 \cdot 3 \cdot 5 \cdot 13$			

[4-64]

11

[4-65]

For

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Answers for Part A.

1. 2,000	2. 2.63	3. 6,210,000
4. 5,300,000	5. 11,400	
6. 4,300,000,000,000,000,000,000		7. 603,800
8. 1,318.9	9. 2,010,000,000,000,000,000,000,000,000,000	

*

Answers for Part B [on pages 4-65 and 4-66].

1. 6.71×10^8	2. 6.595×10^{21}	3. 7.28×10^2
4. 1.657×10^3	5. 4.378×10^6	6. 2.736×10^9
7. 5.7646×10^3	8. 3.754×10^2	9. 4.234×10^6
10. 2.376×10^5	11. 7.832×10^3	12. 5.278×10^8
13. 7.51×10^3	14. 5.46×10^4	15. 8.5×10^5
16. 7.42×10^3	17. 5.46×10^2	18. 5.46×10
19. 5.46	20. $\frac{5.46}{10}$	

[The answers for Exercises 18, 19, and 20 are, of course, <u>not</u> scientific notation. But, they are as close as most students can come at this time. Read on in the text!]

SCIENTIFIC NOTATION

Scientists and engineers often deal with very large numbers. For example, the distance between the Earth and the Sun is about 93,000,000 miles, the speed of light is about 671,000,000 miles per hour, and the mass of the Earth is about 6,595,000,000,000,000,000,000 tons. The three numerals which you have just seen are awkward to write and difficult to read. For these reasons, scientists use a more compact notation. You will learn about this scientific notation in the exercises which follow.

EXERCISES

__A__. Write an equivalent numeral without using exponents, parentheses, or multiplication signs.

1. 2×10^3	2. 0.00263×10^3	3. 6.21×10^6
4. 53×10^5	5. 1.14×10^4	6. 4.3×10^{21}
7. 6.038×10^5	8. 1.3189×10^3	9. 2.01×10^{30}

* * *

A number is named in scientific notation by expressing it as:

(a number between 1 and 10) $\times$ (a power of 10).

For example, the distance in miles between the Earth and the Sun is given in scientific notation by:

$$9.3 \times 10^7.$$

Most of the numbers listed in Part A are listed in scientific notation. Which of them are not?

* * *

__B__. Express in scientific notation.

__Sample 1__. 276.9

__Solution__. $276.9 = 2.769 \times 10^2$

1. the speed of light in miles per hour

(continued on next page)

2. the mass, in tons, of the Earth

3. 728

4. 1657

5. 4, 378, 000

6. 2, 736, 000, 000

7. 5764.6

8. 375.4

<u>Sample</u> <u>2</u>. 67.5×10^3

<u>Solution</u>. $$67.5 \times 10^3 = (6.75 \times 10) \times 10^3$$
$$= 6.75 \times 10^4.$$

9. 423.4×10^4

10. 2376×10^2

11. 78.32×10^2

12. $(58 \times 10^2) \times (91 \times 10^3)$

<u>Sample</u> <u>3</u>. 0.632×10^4

<u>Solution</u>. $$0.632 \times 10^4 = \frac{6.32}{10} \times 10^4$$
$$= 6.32 \times \frac{10^4}{10}$$
$$= 6.32 \times 10^3.$$

13. 0.751×10^4

14. 0.546×10^5

15. 0.085×10^7

16. 0.00742×10^6

17. 0.546×10^3

18. 0.0546×10^3

19. 0.00546×10^3

20. 0.000546×10^3

* * *

When you worked the last two or three exercises in Part B, you may have found it difficult to express each number as the product of

[4-67]

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?

A consequence of:

$$\forall_a \ a^0 = 1$$

is that $0^0 = 1$. In most elementary textbooks, '0^0' is not defined. However, it is convenient to accept the definition that $0^0 = 1$, and, in fact, this is tacitly done in almost all analysis textbooks. [Landau makes explicit mention of this on page 11 of <u>Differential and Integral Calculus</u> (New York: Chelsea, 1951).]

*

Answers for Part C.

1. 2.73×10^1
2. 2.73×10^0
3. 2.73×10^{-1}
4. 2.73×10^{-2}
5. 2.73×10^{-3}
6. 2.73×10^{-4}
7. 7.54×10^{-1}
8. 1.62×10^{-9}

*

Answers for Part D [on page 4-68].

1. 10^7
2. 10^{-5}
3. 10^1
4. 10^2
5. 10^8
6. 10^{-10}
7. 10^{-1}
8. 10^{-7}
9. 10^{-9}
10. 10^6

*

Answers for Part E [on page 4-69].

1. 2.16×10^4
2. 2.04×10^2
3. 2.8×10^3
4. 4.8×10^{-11}
5. 10^2
6. 3.2×10^{-9}
7. 4.9×10^{-5}
8. 7.29×10^{14}
9. 6.4×10^{-7}

a number between 1 and 10 and a power of 10. Let us reconsider Exercises 17-20 of Part B.

$$0.546 \times 10^3 = \frac{5.46}{10} \times 10^3 = 5.46 \times \frac{10^3}{10} = 5.46 \times 10^2$$

$$0.0546 \times 10^3 = \frac{5.46}{10^2} \times 10^3 = 5.46 \times \frac{10^3}{10^2} = 5.46 \times 10$$

$$0.00546 \times 10^3 = \frac{5.46}{10^3} \times 10^3 = 5.46 \times \frac{10^3}{10^3} = 5.46 \times 1$$

$$0.000546 \times 10^3 = \frac{5.46}{10^4} \times 10^3 = 5.46 \times \frac{10^3}{10^4} = 5.46 \times \frac{1}{10}$$

These results suggest that we write '10^1' for '10', '10^0' for '1', and '10^{-1}' for '$\frac{1}{10}$'. In general,

for each real number a, $a^1 = a$, and $a^0 = 1$, and

for each real number $a \neq 0$, for each positive integer n,

$$a^{-n} = \frac{1}{a^n}.$$

In particular,

$$0.0546 \times 10^3 = 5.46 \times 10^1,$$

$$0.00546 \times 10^3 = 5.46 \times 10^0, \text{ and}$$

$$0.000546 \times 10^3 = 5.46 \times 10^{-1}.$$

So, among the powers of 10 we include 10 [which is 10^1], 1[which is 10^0], $\frac{1}{10}$[which is 10^{-1}], $\frac{1}{100}$[which is 10^{-2}], etc. [Read the new numerals as 'the first power of 10', 'the zeroth power of 10', 'the negative first power of 10', etc.]

* * *

C. For each number listed below, write its name in scientific notation.

1. 27.3
2. 2.73
3. 0.273
4. 0.0273
5. 0.00273
6. 0.000273
7. 0.754
8. 0.0000000162

D. Simplify.

Sample 1. $10^3 \times 10^7 \times 10^{-6}$

Solution. $10^3 \times 10^7 \times 10^{-6}$

$$= 10^{10} \times 10^{-6}$$

$$= 10^{10} \times \frac{1}{10^6}$$

$$= \frac{10^{10}}{10^6}$$

$$= 10^4.$$

Sample 2. $\dfrac{10^5 \times 10^{-2}}{10^{-8}}$

Solution. $\dfrac{10^5 \times 10^{-2}}{10^{-8}}$

$$= 10^5 \times \frac{1}{10^2} \times \frac{1}{\frac{1}{10^8}}$$

$$= 10^5 \times \frac{1}{10^2} \times 10^8$$

$$= \frac{10^5 \times 10^8}{10^2}$$

$$= 10^{11}.$$

1. $10^8 \times 10^3 \times 10^{-4}$
2. $10^{-3} \times 10^{-1} \times 10^5 \times 10^{-6}$
3. $10^0 \times 10^{-2} \times 10^3$
4. $10^7 \times 10^{-2} \times 10^{-3}$
5. $10^6 \div 10^{-2}$
6. $10^{-7} \div 10^3$
7. $\dfrac{10^4 \times 10^{-3}}{10^2}$
8. $\dfrac{10^5 \times 10^{-9}}{10^3}$
9. $\dfrac{10^{-7} \times 10^{-3}}{10^{-7} \times 10^6}$
10. $\dfrac{10^4 \times 10 \times 10^{-3}}{10^{-5} \times 10^2 \times 10^{-1}}$

E. Simplify, and use scientific notation for the results.

Sample 1. $9800000000 \times 0.00025$

Solution. $9800000000 \times 0.00025$

$$(9.8 \times 10^9) \times (2.5 \times 10^{-4})$$

$$= (9.8 \times 2.5) \times (10^9 \times 10^{-4})$$

$$= 24.5 \times 10^5$$

$$= 2.45 \times 10^6 .$$

Sample 2. $\dfrac{(65 \times 10^5) \times (9 \times 10^{-2})}{(15 \times 10^4) \times (26 \times 10^{-4})}$

Solution. $\dfrac{(65 \times 10^5) \times (9 \times 10^{-2})}{(15 \times 10^4) \times (26 \times 10^{-4})}$

$$\frac{(65 \times 9) \times (10^5 \times 10^{-2})}{(15 \times 26) \times (10^4 \times 10^{-4})}$$

$$= \frac{\overset{\overset{1}{\cancel{13}}}{\cancel{65}} \times \overset{3}{\cancel{9}}}{\underset{\underset{1}{\cancel{3}}}{\cancel{15}} \times \underset{2}{\cancel{26}}} \times \frac{10^3}{10^0}$$

$$= 1.5 \times 10^3 .$$

1. $7200 \times 1500 \times 0.002$

2. $(68 \times 10^4) \times 0.0003$

3. $(4 \times 10^5) \times (7 \times 10^{-3})$

4. $(6 \times 10^{-7}) \times (8 \times 10^{-5})$

5. $\dfrac{(9 \times 10^{-2}) \times (24 \times 10^3)}{(8 \times 10^{-5}) \times (27 \times 10^4)}$

6. $\dfrac{(81 \times 10^{-5}) \times (64 \times 10^{-2})}{(54 \times 10^{-3}) \times (30 \times 10^5)}$

7. $(7 \times 10^{-3})^2$

8. $(9 \times 10^4)^3$

9. $(0.0008)^2$

F. Solve these problems.

1. Is $10^{15} + 10^{2}$ closer to 10^{15} or closer to 10^{17} ?

2. Is 0.000578 closer to 10^{-4} than to 10^{-3}? Than to 10^{-5}?

3. A movie is shown at the rate of 24 frames per second. The number of individual frames needed for a film which lasts 90 minutes is closest to which of these numbers?

   (a) 10^{3} (b) 10^{4} (c) 10^{5} (d) 10^{6}

4. A watch ticks 5 times each second. The number of times it ticks in one year is closest to which of these numbers?

   (a) 10^{6} (b) 10^{7} (c) 10^{8} (d) 10^{9}

5. The human heart beats day and night 60 times per minute, on the average. The number of times it beats during a 70-year lifespan is closest to which of these numbers?

   (a) 10^{8} (b) 10^{10} (c) 10^{12} (d) 10^{14}

6. Light travels 3×10^{8} meters each second. How many centimeters does it travel in 1 hour? [1 centimeter = 10^{-2} meters.]

7. A .30-caliber bullet takes about 10^{-1} seconds to travel 100 meters, and a fly can beat its wings once in about 10^{-3} seconds. How many times can a fly beat its wings during the time it takes a .30-caliber bullet to travel 100 meters?

8. The length of a diameter of a red blood corpuscle is about 10^{-5} meters and the average distance between the Earth and the Moon is about 10^{8} meters. About how many red blood corpuscles would it take to make a chain of them which stretched from the Earth to the Moon?

The problems in Part F are based on information in Volume 1 of Physics, Preliminary Edition (Cambridge, Massachusetts: Physical Science Study Committee, 1957). They illustrate the importance of scientific notation in making estimates. We recommend that you consult this text for more problems of this type.

*

Answers for Part F.

1. One of the purposes of this exercise is to convince the students, once and for all, that $10^{15} + 10^2 \neq 10^{17}$. Adding 100 is a lot different than multiplying by 100. Since

$$\begin{aligned} 10^{17} &= 10^{15} \times 10^2 \\ &= 10^{15}(1 + 99) \\ &= 10^{15} + (10^{15} \times 99), \end{aligned}$$

and $$10^2 < (10^{15} \times 99)/2,$$

it follows that $10^{15} + 10^2$ is closer to 10^{15} than it is to 10^{17}.

It is instructive to handle this problem graphically. Draw a picture [uniform scale] of the number line on the blackboard and mark on it the graphs of 10^{15} and 10^{16}, making the marks 5 inches apart.

Next, ask for a volunteer to make a mark on this picture for the graph of 10^{17}. It is 50 inches to the right of the graph of 10^{16}!

The distance between 10^{15} and 10^{16} is $|10^{16} - 10^{15}|$, or 9×10^{15}. The distance between 10^{16} and 10^{17} is 9×10^{16}. The distance between 10^{15} and $10^{15} + 10^2$ is 10^2. So, the graph of $10^{15} + 10^2$ is a dot which is

$$\frac{10^2}{9 \times 10^{15}} \times 5 \text{ inches}$$

to the right of the graph of 10^{15}. ["$10^{15} + 10^2$ is very nearly 10^{15} and not at all close to 10^{17}!"]. Finally, ask where you should mark the graph of 0 on your picture of the number line. [Answer: 5/9 of an inch to the left of the graph of 10^{15}.]

[4-70]

F.

2. [The purpose of this exercise [and of Exercises 3 and 4] is to give the student a brief encounter with the concept of order of magnitude. The order of magnitude of a number is the power of 10 to which it is closest.]

   Since $0.000578 = 5.78 \times 10^{-4}$, and since $10^{-3} = 10 \times 10^{-4}$,

   $$10^{-4} < 0.000578 < 10^{-3}.$$

   Since the midpoint of the segment $\overline{10^{-4},\ 10^{-3}}$ is

   $$\frac{10^{-4} + 10^{-3}}{2} = \frac{10^{-4}(1 + 10)}{2} = 5.5 \times 10^{-4},$$

   it follows that 5.78×10^{-4} is closer to 10^{-3} than to 10^{-4}.

   [So, the answer to the first question in the exercise is 'no'.] Since $10^{-5} < 10^{-4} < 5.78 \times 10^{-4}$, 5.78×10^{-4} is closer to 10^{-4} than to 10^{-5}. [So, the answer to the second question is 'yes'.] [The order of magnitude of 5.78×10^{-4} is 10^{-3}.]

   It may help in Exercise 2 to make a number line picture containing graphs of 10^{-5}, 10^{-4}, 5.78×10^{-4}, and 10^{-3}.

3. $$\begin{aligned} 24 \times (90 \times 60) &= 2.4 \times 10^1 \times 9 \times 10^1 \times 6 \times 10^1 \\ &= 2.4 \times 5.4 \times 10^4 \\ &= 12.96 \times 10^4 \\ &= 1.296 \times 10^5. \end{aligned}$$

   Since $10^5 < 1.296 \times 10^5 < 10^6$, and since the midpoint of $\overline{10^5,\ 10^6}$ is 5.5×10^5, the order of magnitude of 1.296×10^5 is 10^5. So, the answer is: (c) 10^5.

4. $$\begin{aligned} &5 \times 365 \times 24 \times 60 \times 60 \\ &= 5 \times 3.65 \times 10^2 \times 2.4 \times 10^1 \times 6 \times 10^1 \times 6 \times 10^1 \\ &= 5 \times 3.65 \times 2.4 \times 3.6 \times 10^6. \end{aligned}$$

   This number is less than $5 \times 4 \times 3 \times 4 \times 10^6$, or 2.4×10^8. Also, it is greater than $5 \times 3.5 \times 2 \times 3 \times 10^6$, or 1.05×10^8. So, its order of magnitude is 10^8, and the answer for the exercise is: (c) 10^8.

[4-70]

F

5.
$$\begin{aligned} & 70 \times 365 \times 24 \times 60 \times 60 \\ = \; & 7 \times 3.65 \times 2.4 \times 6 \times 6 \times 10^6 \\ < \; & 7 \times 4 \times 3 \times 6 \times 6 \times 10^6 \\ = \; & 84 \times 36 \times 10^6 \\ < \; & 3.6 \times 10^9. \end{aligned}$$

So, the number in question is less than 3.6×10^9.

$$\begin{aligned} & 70 \times 365 \times 24 \times 60 \times 60 \\ = \; & 7 \times 3.65 \times 2.4 \times 6 \times 6 \times 10^6 \\ > \; & 7 \times 3 \times 2 \times 6 \times 6 \times 10^6 \\ = \; & 42 \times 36 \times 10^6 \\ > \; & 1.2 \times 10^9. \end{aligned}$$

So, the number in question is greater than 1.2×10^9. [Since the number in question is between 10^9 and the midpoint (5.5×10^9) of the interval from 10^9 to 10^{10}, its order of magnitude is 10^9.] 10^9 is not one of the numbers listed, but the choice has been narrowed down to that between 10^8 and 10^{10}. If N is the number in question then

$$N - 10^8 < 3.6 \times 10^9 - 10^8 = 3.5 \times 10^9, \text{ and}$$
$$10^{10} - N > 10^{10} - 3.6 \times 10^9 = 6.4 \times 10^9.$$

So, since $10^8 < N < 3.6 \times 10^9$ and since 3.6×10^9 is closer to 10^8 than to 10^{10}, it follows that N is closer to 10^8 than to 10^{10}. The answer for the exercise is: (a) 10^8.

6. $3 \times 10^8 \times 10^2 \times 60 \times 60 = 3 \times 36 \times 10^{12} = 1.08 \times 10^{14}$; 1.08×10^{14} centimeters per hour.

7. $\dfrac{10^{-1}}{10^{-3}} = \dfrac{1}{10^1} \div \dfrac{1}{10^3} = \dfrac{10^3}{10^1} = 10^2$; 100 times.

8. $\dfrac{10^8}{10^{-5}} = 10^8 \times 10^5 = 10^{13}$; 10^{13} corpuscles.

*

Summary of the answers for Part F.

1. 10^{15}	2. No; yes	3. (c) 10^5	4. (c) 10^8
5. (a) 10^8	6. 1.08×10^{14}	7. 100	8. 10^{13}

[4-70]

F

[4-71]

The purpose of these Exploration Exercises is to prepare the way for the notions of highest common factor and lowest common multiple. Students will obtain maximum benefit from Parts A and B if, as in the Sample, they use the prime power factors of a number in finding numbers which are factors of it.

*

Answers for Part A.

1. {1, 2, 3, 4, 6, 12}, {1, 2, 3, 6, 9, 18}; {1, 2, 3, 6}
2. {1, 2, 5, 10, 25, 50}, {1, 3, 5, 15, 25, 75}, {1, 2, 3, 5, 6, 9, 10, 15, 18, 30, 45, 90}; {1, 5}
3. {1, 2, 4, 8}, {1, 2, 4, 8, 16}, {1, 2, 4, 5, 8, 10, 20, 40}; {1, 2, 4, 8}
4. {1, 3}, {1, 5}, {1, 7}; {1}

*

Answers for Part B.

1. {1, 2, 3, 6} 2. {1, 2, 3, 6} 3. {1, 2, 4} 4. {1, 3, 9}
5. {1, 2, 5, 10} 6. {1, 2} 7. {1, 2, 4} 8. {1, 3}

*

Answers for questions at top of page 4-72.

There is just one multiple of 0 with respect to the set of real numbers.

There are as many multiples of 7 with respect to the set of real numbers as there are real numbers. [Since $7 \neq 0$, the left cancellation principle for multiplication tells us that different real numbers used as multipliers yield different multiples of 7.]

There are as many multiples of 2 with respect to the set of positive integers as there are positive integers.

*

Answers for Part C [on page 4-72].

1. 3, 6, 9, 12, 15, 18 2. 56, 63, 70, 77, 84, 91, 98
3. 24 4. 102 5. 6, 12 6. 4×25
7. 10×21 8. $2 \times 3 \times 5$, $2 \times 2 \times 3 \times 5$, 6×15 9. b, c, d, e, f, h

EXPLORATION EXERCISES

A. In each exercise you are given two or more numbers. For each number in the exercise, find the set of its factors. Then, find the intersection of these sets.

Sample. 36, 45, 90

Solution. $36 = 3^2 \cdot 2^2$, $45 = 3^2 \cdot 5$, $90 = 3^2 \cdot 2 \cdot 5$.

The factor-set of 36 is $\{1, 3, 9, 2, 6, 18, 4, 12, 36\}$, the factor-set of 45 is $\{1, 5, 3, 15, 9, 45\}$, and the factor-set of 90 is $\{1, 5, 3, 15, 9, 45, 2, 10, 6, 30, 18, 90\}$. The intersection of these factor-sets is $\{1, 3, 9\}$.

1. 12, 18 2. 50, 75, 90 3. 8, 16, 40 4. 3, 5, 7

B. For each exercise, find the intersection of the factor-sets of the numbers listed. [That is, find the set of common factors of the numbers listed in the exercise.]

1. 18, 30
2. 12, 30, 42
3. 16, 36, 72
4. 9, 90, 900
5. 20, 30, 40
6. 48, 14, 28
7. 64, 108, 300
8. 12, 33, 66, 132

* * *

7 is a factor of 21 and 21 is a multiple of 7. 36 is a multiple of 4 and a multiple of 3. A first number is a multiple of a second number if and only if the second number is a factor of the first. So, the notion of multiple, like the notion of factor, depends upon the set of numbers under discussion. For example, 7 is a multiple of 3 with respect to the set of rationals but not with respect to the set of positive integers.

Some multiples of 5 with respect to the set of positive integers are

5, 10, 15, 20, 25, and 30.

What are four more such multiples of 5? Three multiples of 7 with respect to the set of integers are

$^{-}14$, 0, and 700.

Of course, each real number is a multiple of each nonzero real number with respect to the set of real numbers. How many multiples of 0 are there with respect to the set of real numbers? How many multiples of 7 are there with respect to the set of reals? How many multiples of 2 are there with respect to the set of positive integers?

As we agreed earlier for the word 'factor', unless otherwise specified, when we speak of multiples of numbers, we shall mean multiples with respect to the set of positive integers.

* * *

C. 1. What are the multiples of 3 which are less than 20?

2. What are the multiples of 7 between 50 and 100?

3. How many multiples of 4 are there between 201 and 299?

4. How many multiples of 13 are there between 7241 and 8573?

5. What numbers less than or equal to 15 are <u>common multiples</u> of 2 and 3, that is, numbers which are multiples of both 2 and 3?

6. What numbers less than or equal to 4×25 are common multiples of 4 and 25?

7. What numbers less than or equal to 10×21 are common multiples of 10 and 21?

8. What numbers less than or equal to 6×15 are common multiples of 6 and 15?

9. Each of the following exercises lists a pair of numbers. Draw a loop around [or copy the letter of] each exercise for which the numbers have a common multiple which is smaller than their product.

(a) 3, 5	(b) 8, 12	(c) 4, 8	(d) 10, 12
(e) 10, 24	(f) 6, 9	(g) 9, 20	(h) 20, 30

om-

Of co

be

Since it is a commonplace that students frequently seem unable to think of using definitions in proofs, the proof under discussion is a valuable example.

*

Some students may discover the following proof for Exercise 5(a), which depends upon the generalization that for each set of positive integers a, b, c, ... , n, x is their HCF if and only if there are positive integers a′, b′, c′, ... , n′ such that

$$a = a'x, \quad b = b'x, \quad c = c'x, \quad \ldots, \quad n = n'x,$$

and such that a′, b′, c′, ... , n′ are relatively prime. [This generalization is intuitively clear, but its proof is not easy, and we shall not give it here.]

Suppose that x is the HCF of a, b, and a + b. Then [by the generalization stated above] there are positive integers m, n, and p such that

$$a = mx, \quad b = nx, \quad \text{and} \quad a + b = px,$$

and such that m, n, and p are relatively prime. We shall have proved our theorem if we can show that m and n are relatively prime. Let us suppose that they are not. Then, there is a positive integer $y > 1$ such that y is a factor of m and n. But, since $mx + nx = px$, it follows that $m + n = p$. So, by the theorem of Exercise 1(b) of Part B on page 4-51, y is a factor of p. Hence, y is a factor of m, n, and p. So, m, n, and p are not relatively prime, and this contradicts the supposition that they are relatively prime. Therefore, m and n are relatively prime, and it follows from this [and the generalization stated above], that x is the HCF of a and b.

[4-73]

om-

Of c
b

readily from (*). Here is its proof. Suppose m is a factor n. Then, there is a positive integer k such that $mk = n$. Now, since $m > 0$ and $1 \leq k$, $m \leq mk$. So, $m \leq n$. Similarly, supposing that n is a factor of m, $n \leq m$. So, by (*), since $m \leq n$ and $n \leq m$, it follows that $m = n$.]

(a) One uses (**) in proving the theorem in Exercise 5(a) in this manner. Suppose m is the HCF of a and b, and n is the HCF of a, b, and $a + b$. We show, first, that m is a factor of n and, second, that n is a factor of m. Then, by (**), we conclude that $m = n$. Here, then, is the proof.

By the theorem of Exercise 1(b) of Part B on page 4-51, each common factor of [the positive integers] a and b is a factor of $a + b$. Consequently, the HCF of a and b is a common factor of a, b, and $a + b$. Hence, by the definition of 'HCF', it is a factor of the HCF of a, b, and $a + b$. [Next, we show that the HCF of a, b, and $a + b$ is a factor of the HCF of a and b.] Obviously, each common factor of a, b, and $a + b$ is a common factor of a and b. So, the HCF of a, b, and $a + b$ is a common factor of a and b. Hence, by the definition of 'HCF', it is a factor of the HCF of a and b. Since the HCF of a and b is a factor of the HCF of a, b, and $a + b$, and since the HCF of a, b, and $a + b$ is a factor of the HCF of a and b, it follows [from (**)] that the HCF of a and b is the HCF of a, b, and $a + b$.

(b) The proof of the theorem in Exercise 5(b) is entirely similar to the proof just given except that instead of using the theorem of Exercise 1(b) on page 4-51, we use the analogous theorem that for positive integers a and b, each common factor of a and b is a factor of $85a + 370b$, [and the proof of <u>this</u> theorem should cause no difficulty].

(c) For positive integers p and q, for positive integers a and b,

the HCF of a and b = the HCF of a, b, and $pa + qb$.

[The proof follows the same lines as that for Exercise 5(b).]

As mentioned above, this method of proof is of considerable importance in mathematics. Theorems (*) and (**) are widely applicable. Also, the proof given above for the theorem of Exercise 5(a) depends directly upon the definition of 'HCF'.

[4-73]

om-

Of c
b

Answers for Part A [on pages 4-73 and 4-74].

1. 10   2. 5   3. 1   4. 20   5. 9   6. 14

[If students have been using prime power factorizations in finding common factors, they should by now have found the usual short cut for finding HCFs. If they have not found the short cut, the Sample at the top of page 4-74 should call it to their attention.]

7. 3^2 8. $2^2 \cdot 3^2$ 9. $2^2 \cdot 3^2 \cdot 5$

10. 3   11. 3   12. 1

*

Answers for Part B [on page 4-74].

1. 143/368   2. 15/22   3. 2184/203375

[By now, students will have discovered that the grade school problem of reducing a fraction to lowest terms involves nothing more than dividing numerator-number and denominator-number by their highest common factor. If the numerator-number and the denominator-number of a fraction are positive integers and their HCF is 1, we say that the fraction has been reduced to lowest terms. [Two positive integers whose HCF is 1 are said to be <u>relatively prime</u>.]]

4. yes; yes; yes

☆5. You have here an excellent opportunity to acquaint students with an important technique for deriving a sentence of the form: $x = y$. [This is a technique analogous to one used in showing that a first set A is the same as a second set B. One simply shows that $A \subseteq B$ and $B \subseteq A$, and then concludes from this that $A = B$.] To show that a first number x is the same as a second number y, it is sufficient to show that $x \leq y$ and $y \leq x$. This is merely an application of the theorem:

$$(*) \quad \forall_x \forall_y \text{ if } x \leq y \text{ and } y \leq x \text{ then } x = y.$$

A theorem about <u>positive</u> integers which is closely related to (*) and which is useful in proving the theorem in Exercise 5(a) is:

$$(**) \quad \forall_m \forall_n \text{ if m is a factor of n and n is a factor of m then } m = n.$$

(**) is the theorem you might tell students to use. [(**) follows

om-

Of c

b

10. b, c, d, e, f, h

11. Yes [because they have a common factor greater than 1].

12. a, c, d, f

*

Notice the definition of 'HCF'. The HCF of the members of a [non-empty] set of positive integers is the positive integer x such that

(a) x is a common factor of the members of the given set, and
(b) x is a multiple of each common factor of the members of the given set.

One purpose of the preceding Exploration Exercises is to convince students that there is such a positive integer.

You may be acquainted with an alternative definition for 'HCF' in which condition (b) is replaced by:

(b′) x is greater than or equal to each common factor of the members of the given set.

We have chosen (b) in preference to (b′) because our definition has a natural analogue in the case of pronumeral expressions [see page 4-80]. This is not the case with (b′).

Having chosen (b), we choose to speak of the <u>highest</u> common factor ["largest in the sense of multiplication"] rather than the greatest common factor ["largest in the sense of addition"]. Similarly, we speak of the <u>lowest</u> (rather than of the <u>least</u>) common multiple.

You may be interested to note the similarities between the relation of being greater than or equal to and the relation of being a multiple of. If we use 'M' as an abbreviation for 'is a multiple of' then, for all positive integers a, b, and c,

if $a \geq b$ and $b \geq c$ then $a \geq c$,	if aMb and bMc then aMc;
if $a \geq b$ and $b \geq a$ then $a = b$,	if aMb and bMa then a = b;
$a \geq a$,	aMa ;
$a \geq b$ or $b \geq a$,	________________ .

The similarity breaks down in the last case; it is not the case, for example, that either 2M3 or 3M2. Neither 2 nor 3 is a (positive integral) multiple of the other.

*

10. For the pairs given in Exercise 9, tell which have a common factor greater than 1.

11. Do the numbers 856254 and 739876 have a common multiple less than their product?

12. Draw a loop around [or copy the letter of] each exercise whose numbers have a common multiple smaller than their product.

(a) $3 \cdot 5,\ 3 \cdot 7$	(b) $2 \cdot 5,\ 3 \cdot 7$
(c) $2^3 \cdot 5,\ 2^2 \cdot 3$	(d) $3^2 \cdot 5,\ 3^2 \cdot 5 \cdot 7$
(e) $2^3 \cdot 11,\ 3^2 \cdot 13$	(f) $2^2 \cdot 3 \cdot 5^2,\ 3 \cdot 7 \cdot 13^2$

4.06 Factoring. --Consider the factor-sets of 18 and 30, that is, the sets

$$\{1, 3, 9, 2, 6, 18\} \quad \text{and} \quad \{1, 5, 3, 15, 2, 10, 6, 30\}.$$

Do you see that the common factors of 18 and 30 are 1, 3, 2, and 6? Notice that one of these common factors is a multiple of each of them [Which one?]. Use the results of Parts A and B of the Exploration Exercises to check the generalization that among the common factors of two or more positive integers there is one which is a multiple of each common factor. This common factor is called the highest common factor [HCF, for short] of the given positive integers. [Sometimes the highest common factor is called the greatest common divisor [GCD, for short] because it is the largest positive integer by which each of the integers can be divided "exactly".]

EXERCISES

A. For each exercise, find the HCF of the numbers listed.

1. 20, 30	2. 25, 55	3. 7, 11
4. 100, 120	5. 36, 54, 99	6. 42, 70, 140

(continued on next page)

Sample. 900, 6240

Solution. $900 = 2^2 \cdot 3^2 \cdot 5^2$,

$6240 = 2^5 \cdot 3 \cdot 5 \cdot 13$

The HCF of 900 and 6240 is $2^2 \cdot 3 \cdot 5$.

7. 36, 135 8. 108, 144 9. 900, 1080

10. 36, 42, 105 11. 84, 45, 105 12. 35, 36

B. 1. Use the fact that the HCF of 4147 and 10672 is 29 to reduce to lowest terms the fraction: 4147/10672.

2. Use the fact that the GCD of 630 and 924 is 42 to reduce to lowest terms the fraction: 630/924.

3. Use the fact that the HCF of 2184 and 203375 is 1 to reduce to lowest terms the fraction: 2184/203375.

4. Is the HCF of 120 and 180 the same as the HCF of 120 and 120 + 180? 180 and 120 + 180? 120, 180, and 120 + 180?

☆5. (a) The last result of Exercise 4 suggests the theorem:

$\forall_a \forall_b$ the HCF of a and b = the HCF of a, b, and a + b.

Prove this theorem.

(b) Prove:

$\forall_a \forall_b$ the HCF of a and b = the HCF of a, b, and 85a + 370b.

(c) Generalize the results of (a) and (b).

* * *

You have seen that two or more positive integers have a common factor which is a multiple of each of their common factors.

It is also true that two or more positive integers have a common multiple which is a factor of each of their common multiples.

For example, some of the common multiples of 6 and 8 are 24, 48, 72, 96, and 120. One of these common multiples--24--is a factor of each of them. And, this number is called the lowest common multiple of 6 and 8. [Sometimes it is called the least common multiple because it is the smallest positive integer which is a multiple of both 6 and 8.]

[4-75]

The proof involves the notion of the prime power factorization for positive integers. [See page 4-64.] In general, if p is the largest prime which is a factor of either of the positive integers a and b, then there are nonnegative integers α_2, α_3, α_5, α_7, α_{11}, ..., α_p and β_2, β_3, β_5, β_7, β_{11}, ..., β_p such that

$$a = 2^{\alpha_2} \cdot 3^{\alpha_3} \cdot 5^{\alpha_5} \cdot 7^{\alpha_7} \cdot 11^{\alpha_{11}} \cdot \ldots \cdot p^{\alpha_p}$$

and

$$b = 2^{\beta_2} \cdot 3^{\beta_3} \cdot 5^{\beta_5} \cdot 7^{\beta_7} \cdot 11^{\beta_{11}} \cdot \ldots \cdot p^{\beta_p}.$$

The product of a by b, their HCF, their LCM, and the product of their HCF by their LCM each has a similar prime power factorization. Fill in the blanks in the following expression:

$$a \times b = 2^{\_\_\_\_} \cdot 3^{\_\_\_\_} \cdot 5^{\_\_\_\_} \cdot 7^{\_\_\_\_} \cdot \ldots \cdot p^{\_\_\_\_}.$$

Do you see a rule for finding the exponent for each prime named in the prime power factorization of a × b? Suppose you were writing the prime power factorization for the HCF of a and b. How would you determine the exponent for each prime named in that factorization? Similarly, how would you determine the exponent for each prime named in the prime power factorization of the LCM of a and b? In the prime power factorization of the product of the HCF by the LCM?

The generalization of Exercise 2 follows from the fact that for each first number and each second number, the sum of the numbers is the sum of the lesser and the greater. [If the numbers are equal, "each is both the lesser and the greater".]

In the case of the three (or more) numbers [line (g) of the table, and Exercise 3], we get the exponent for each prime named in the factorization of the product by adding the numbers named by three (or more) exponent symbols. But, we get the exponent for each prime named in the factorization of HCF × LCM by adding the numbers named by only two exponent symbols. The rules for writing the prime power factorizations of the product and of HCF × LCM give the same result just if only one of the three (or more) exponent symbols names a nonzero number. That is, just if, of each of two of the three (or more) exponent symbols, at least one names 0. [This justifies the comment which follows the answer for Exercise 3.]

[4-75]

See TC[4-73, 74]a for the reason for preferring 'lowest common multiple' to 'least common multiple'.

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Answers for Part C.

1. $2^2 \cdot 3^2$
2. $2^2 \cdot 3 \cdot 5$
3. $2 \cdot 3 \cdot 5$
4. $2 \cdot 3$
5. $2 \cdot 5 \cdot 7$
6. $2^2 \cdot 3^2 \cdot 5^2$
7. $2 \cdot 5 \cdot 13$
8. $2^3 \cdot 3 \cdot 5$
9. 2^7
10. 3^5
11. $2^6 \cdot 3^3$
12. $2^4 \cdot 3^2$

*

Answers for Part D.

1.	Product	HCF	LCM	HCF × LCM
(a) 15, 20	300	5	60	300
(b) 30, 50	1500	10	150	1500
(c) 12, 18	216	6	36	216
(d) 15, 28	420	1	420	420
(e) 10, 14	140	2	70	140
(f) $2^4 \cdot 3^2$, $2^2 \cdot 3^3$	$2^6 \cdot 3^5$	$2^2 \cdot 3^2$	$2^4 \cdot 3^3$	$2^6 \cdot 3^5$
(g) 3, 12, 21	756	3	84	252

2. $\forall_a \forall_b$ ab = (HCF of a and b) × (LCM of a and b).

☆3. 5, 9, 14 [Or, any three positive integers such that the HCF of each pair is 1.]

*

Discussion of Part D.

Line (f) of the above table when rewritten as:

(f) $2^4 \cdot 3^2$, $2^2 \cdot 3^3$	$2^{4+2} \cdot 3^{2+3}$	$2^2 \cdot 3^2$	$2^4 \cdot 3^3$	$2^{2+4} \cdot 3^{2+3}$

,

should suggest a proof for the generalization given in answer to Exercise 2.

C. For each exercise, find the LCM of the numbers listed.

Sample. 45, 50, 54

Solution. $45 = 3^2 \cdot 5$

$50 = 2 \cdot 5^2$

$54 = 2 \cdot 3^3$

The LCM of 45, 50, and 54 is $2 \cdot 3^3 \cdot 5^2$.

1. 12, 18
2. 4, 6, 15
3. 2, 3, 5
4. 2, 3, 6
5. 10, 14, 35
6. 18, 45, 100
7. 26, 65
8. 6, 8, 20
9. 4, 16, 128
10. $3, 3^2, 3^5$
11. $6^2, 12^3, 1^2$
12. $1^2, 2^2, 3^2, 4^2$

D. 1. Complete this table.

	Product	HCF	LCM	HCF × LCM
(a) 15, 20				
(b) 30, 50				
(c) 12, 18				
(d) 15, 28				
(e) 10, 14				
(f) $2^4 \cdot 3^2$, $2^2 \cdot 3^3$				
(g) 3, 12, 21				

2. State a generalization suggested by the results of Exercise 1.

☆3. Find three numbers whose product is the product of their HCF by their LCM.

FACTORING PRONUMERAL EXPRESSIONS

You have seen that in speaking of factors of a number it is necessary to specify the set of numbers to which the factors are to belong. For example, the number 34 has 1, 2, 17, and 34 as factors with respect to the positive integers. But, with respect to the set of all integers,

it has additional factors--$^-1$, $^-2$, $^-17$, and $^-34$. Also, each nonzero rational number is a factor of 34 with respect to the set of rationals.

Suppose you are asked to factor an expression such as '34'. This means that you are to find an expression "in the form of a product" which is equivalent to '34'. Clearly, there are many such expressions, and each is called <u>a</u> <u>factorization</u> <u>of</u> '<u>34</u>'. For example, one factorization of '34' is its prime factorization, '$2 \cdot 17$'. If we decide that the factors occurring in the factorization are to be names for positive integers then '$2 \cdot 17$' is a "complete" factorization of '34'. [You can't factor any more except by introducing '1's.]

If we decide that the factors occurring in the factorization are to be names for rational numbers then '$\frac{3}{5} \cdot \frac{85}{8} \cdot \frac{48}{9}$' is a factorization of '34'. It is clear that, under these conditions, there is no such thing as a complete factorization of '34'. [There are no numbers which are "prime" with respect to the set of rationals.] When we allow numerals for irrational numbers to occur as factors, we get other factorizations of '34', such as '$\sqrt{2} \cdot \sqrt{34} \cdot \sqrt{17}$'.

If you are asked to factor an expression which is a name for a positive integer, you should assume that what is wanted is the prime factorization [or the prime power factorization] of the positive integer.

Similarly, in order to speak precisely about factors and factorizations of pronumeral expressions, we would have to specify the set of expressions which were to be eligible as factors. [And, it turns out, we would also have to specify the domains of the pronumerals.] Let's take an example. Suppose we decide to use as factors expressions whose only numerals are those for integers. Under this requirement, it is not possible to factor '$2x + 1$' [if we want the domain of 'x' to contain all real numbers]. But, if we allow numerals for rational numbers to occur in factors, one factorization of '$2x + 1$' is '$2(x + \frac{1}{2})$', and another is '$\frac{3}{5}\left(\frac{10}{3}x + \frac{5}{3}\right)$'. And, if we allow restrictions to be put on the domain of 'x', there are additional factorizations; for example, '$x(2 + \frac{1}{x})$'.

As a second example, consider factorizations of '$x^4 - 9$'. This expression is equivalent to '$(x^2 - 3)(x^2 + 3)$'. If we disregard expressions which contain numerals for nonintegral numbers, and require

Beginning at the foot of page 4-75, and continuing through page 4-86, we consider the meaning of 'factor' one has in mind when one says, for example:

factor the expression '$x^3 - xy^2$'.

Students have already become acquainted with this use of 'factor' from pages 3-90ff. They should now be prepared to understand that the ambiguity of 'factor', when applied to expressions, stems from not specifying the set of expressions which are admissible as factors. As pointed out in the middle of page 4-77, we are not yet ready to remove this ambiguity. However, knowing its source should, by itself, dispel some doubts which may have been troubling students. In a later unit on polynomial functions we shall be in a position to be more precise. We can then, for example, speak of factoring with respect to the set of polynomials in 'x'.

*

Some students have difficulty in recognizing, for example, that '$a(a + 3) - 10$' is not a factorization of '$a^2 + 3a - 10$'. In this connection it may be helpful to introduce the notion of the principal operator of an expression. [See TC[1-82]b and TC[1-38, 39].] When all of the grouping symbols have been restored in unabbreviating an expression, the principal operator is the one which corresponds with the pair of outermost grouping symbols. For example, the principal operator in '$a^2 + 3a - 10$' is '$-$' because unabbreviating this expression gives us:

$$\{[(a \times a) + (3 \times a)] - 10\}.$$

A factorization of an expression is an expression in which the principal operator is '$\times$'. Since the principal operator of '$a(a + 3) - 10$' is '$-$', it is not a factorization of '$a^2 + 3a - 10$'. But, '$(a + 5)(a - 2)$' is a factorization of the given expression because they are equivalent, and because the principal operator of '$(a + 5)(a - 2)$' is '$\times$' [omitted by convention].

[An expression whose principal operator is '$+$' is sometimes called <u>an indicated sum</u>, one whose principal operator is '$\times$' is called <u>an indicated product</u>, etc.]

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that the domain of 'x' be the set of real numbers, then this is a "complete" factorization. But, if we allow numerals for arbitrary real numbers to occur in the factors, we can "factor further" to obtain:

$$(x - \sqrt{3})(x + \sqrt{3})(x^2 + 3),$$

or even such a silly thing as:

$$\frac{6}{7}\left(\frac{x}{3} - \frac{\sqrt{3}}{3}\right)\left(\frac{x}{2} + \frac{\sqrt{3}}{2}\right)(7x^2 + 21).$$

And, if we are willing to restrict the values of 'x' to the set of non-negative real numbers, we can factor '$x^4 - 9$' and obtain:

$$(\sqrt{x} - \sqrt{\sqrt{3}})(\sqrt{x} + \sqrt{\sqrt{3}})(x + \sqrt{3})(x^2 + 3).$$

It is not easy [nor would it be profitable here] to describe the possible sets of pronumeral expressions with respect to which one may factor a given expression. We shall content ourselves with two general remarks which you can use as guides in deciding when an expression has been "completely" factored.

First, the factors of an expression should be simpler than the expression itself [or, at any rate, no more complicated than the given expression]. Here, as usual, the meaning of 'simpler than' is vague, and varies from one context to another. But, just as you learned to speak more or less grammatically long before you were aware of grammar, or had any inkling as to its rules, so you must now develop a feeling for the meaning of 'simpler than' in the absence of any rules. One of the reasons for working numerous factoring, expanding, and simplifying exercises is to develop such a feeling. The best we can do is to give you a few examples. Suppose, for instance, that you are merely asked to factor '$4x^2 - 1$'. There are two more or less reasonable choices for an answer, '$(2x - 1)(2x + 1)$' and '$4(x - \frac{1}{2})(x + \frac{1}{2})$'. Now, in the absence of any further instructions, one would probably choose the first of these, perhaps on the ground that '$4x^2 - 1$' contains no fractions, while '$4(x - \frac{1}{2})(x + \frac{1}{2})$' does. On the other hand, if one were factoring '$4x^2 - 1$' for the purpose of reducing the fraction '$\frac{x - \frac{1}{2}}{4x^2 - 1}$', it would be more appropriate to use the factorization '$4(x - \frac{1}{2})(x + \frac{1}{2})$' or the factorization '$2(x - \frac{1}{2})(2x + 1)$'.

Again, if you were asked to factor '$\frac{a^2}{4} - \frac{b^2}{9}$', it would be hard to decide between '$\left(\frac{a}{2} - \frac{b}{3}\right)\left(\frac{a}{2} + \frac{b}{3}\right)$' or '$\frac{1}{36}(3a - 2b)(3a + 2b)$'. But, if you had an ulterior motive for factoring, then one of them [and it could be either] might be much simpler to use than the other.

The second remark has to do with the fact that, in factoring pronumeral expressions, it is generally not important to factor numerals completely. Thus, if asked to factor '$36x^2 + 72$', it is enough to write '$36(x^2 + 2)$'. There is no point, in this context, in writing '$2^2 \cdot 3^2(x^2 + 2)$'.

EXERCISES

A. Factor.

Sample 1. $12xy + 15xz$

Solution. $3x(4y + 5z)$

Sample 2. $18x^2y - 24x^3y^2$

Solution. $6x^2y(3 - 4xy)$

1. $4y + 8x$	2. $6p - 3q$	3. $yc - yd$
4. $12R - 12r$	5. $7y - 21$	6. $8x - 8y$
7. $7x + 7$	8. $4 - 24c$	9. $x^2 - 3x$
10. $4x^2 + 5x$	11. $9y^2 - 18y$	12. $38x + x^2$
13. $cy - 5cb$	14. $10y + 15y^3$	15. $2y - 4y^3$
16. $25x^2y - 35y^2x$	17. $12z^2 + 12$	18. $rt - 5rs$
19. $\pi r^2 + 2\pi rh$	20. $yx^2 + 9y$	21. $7m - 21m^3$
22. $11y^4 + 11y^2$	23. $3p^2 - 39p^3$	24. $bx + bx^3$
25. $2ab^2 + 4a^2b$	26. $10yx - 30x^2y^2$	27. $42m^3n^2 - 14m^2n$
28. $15r^2s^3 - 24s^4r$	29. $72m^2n^2 - 54m^2n$	30. $200tv^2 - 160t^2v$

[Supplementary exercises are in Part N, pages 4-128 and 4-129.]

Answers for Part A [on pages 4-78, 4-79, and 4-80].

1. $4(y + 2x)$ 2. $3(2p - q)$ 3. $y(c - d)$ 4. $12(R - r)$

5. $7(y - 3)$ 6. $8(x - y)$ 7. $7(x + 1)$ 8. $4(1 - 6c)$

9. $x(x - 3)$ 10. $x(4x + 5)$ 11. $9y(y - 2)$ 12. $x(38 + x)$

13. $c(y - 5b)$ 14. $5y(2 + 3y^2)$ 15. $2y(1 - 2y^2)$

16. $5xy(5x - 7y)$ 17. $12(z^2 + 1)$ 18. $r(t - 5s)$

19. $\pi r(r + 2h)$ 20. $y(x^2 + 9)$ 21. $7m(1 - 3m^2)$

22. $11y^2(y^2 + 1)$ 23. $3p^2(1 - 13p)$ 24. $bx(1 + x^2)$

25. $2ab(b + 2a)$ 26. $10xy(1 - 3xy)$ 27. $14m^2n(3mn - 1)$

28. $3rs^3(5r - 8s)$ 29. $18m^2n(4n - 3)$ 30. $40tv(5v - 4t)$

[As has been pointed out in a similar case [see line 5 on page 4-77], the expressions given as answers for Exercises 15 and 21 "can" be factored further to yield '$2y(1 - \sqrt{2}y)(1 + \sqrt{2}y)$' and '$7m(1 - \sqrt{3}m)(1 + \sqrt{3}m)$'. Students who notice this should be complimented.]

31. $(t - 2)(t + 2)$ 32. $(m - 6)(m + 6)$ 33. $(10 - k)(10 + k)$

34. $(2 - s)(2 + s)(4 + s^2)$ 35. $(s^2 - 7)(s^2 + 7)$

36. $(c - 0.4)(c + 0.4)$ 37. $(2m - 11)(2m + 11)$

38. $(p - 3)(p + 3)(p^2 + 9)$ 39. $[m - (1/8)][m + (1/8)]$

40. $[t - (1/10)][t + (1/10)]$ 41. $[7m - (1/4)][7m + (1/4)]$

42. $[(3/5) - 9d][(3/5) + 9d]$ 43. $(x^2 - y)(x^2 + y)$

44. $(2m^2 - 3n)(2m^2 + 3n)$ 45. $[6p^2 - (1/7)][6p^2 + (1/7)]$

46. $(y + 8)(y + 1)$ 47. $(m - 2)(m - 4)$ 48. $3(a - 2)(a - 3)$

49. $7(x + 2)(x - 5)$ 50. $2(18 + x)(2 - x)$ 51. $m(6 + m)(7 - m)$

52. $2x(3x - 1)(x - 5)$ 53. $10y^2(7y + 1)(y - 2)$ 54. $5(a + 1)(a + 2)$

55. $3c(c + 5)(c + 1)$ 56. $10(x + 7)(x + 1)$ 57. $y^4(y + 9)(y + 1)$

58. $m(m + 7t + 10)$ 59. $5(x - 1)(x + 1)$

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<u>Sample 3.</u> $x^2 - y^2$

<u>Solution.</u> $(x - y)(x + y)$

<u>Sample 4.</u> $16a^4 - b^4$

<u>Solution.</u> $(4a^2)^2 - (b^2)^2$

$= (4a^2 - b^2)(4a^2 + b^2)$

$= (2a - b)(2a + b)(4a^2 + b^2)$.

31. $t^2 - 4$
32. $m^2 - 36$
33. $100 - k^2$
34. $16 - s^4$
35. $s^4 - 49$
36. $c^2 - 0.16$
37. $4m^2 - 121$
38. $p^4 - 81$
39. $m^2 - (1/64)$
40. $t^2 - (1/100)$
41. $49m^2 - (1/16)$
42. $(9/25) - 81d^2$
43. $x^4 - y^2$
44. $4m^4 - 9n^2$
45. $36p^4 - (1/49)$

[Supplementary exercises are in Part N on page 4-129.]

<u>Sample 5.</u> $a^2 + 8a + 12$

<u>Solution.</u> $(a + 6)(a + 2)$

<u>Sample 6.</u> $c^4 + c^3 - 6c^2$

<u>Solution.</u> $c^2(c^2 + c - 6)$

$= c^2(c + 3)(c - 2)$.

46. $y^2 + 9y + 8$
47. $m^2 - 6m + 8$
48. $3a^2 - 15a + 18$
49. $7x^2 - 21x - 70$
50. $72 - 32x - 2x^2$
51. $42m + m^2 - m^3$
52. $6x^3 - 32x^2 + 10x$
53. $70y^4 - 130y^3 - 20y^2$
54. $5a^2 + 15a + 10$
55. $3c^3 + 15c + 18c^2$
56. $80x + 10x^2 + 70$
57. $y^6 + 10y^5 + 9y^4$
58. $m^2 + 7mt + 10m$
59. $5x^2 - 5$

(continued on next page)

60. $8x^2 + x^4 + 6x^3$

61. $x^4 - 5x^2 + 4$

62. $y^4 - 8y^2 - 9$

63. $6x^3y^2 + 5x^2y^2 - 4xy^2$

64. $u^6 - 9y^6$

65. $(x + y)^2 - (x - y)^2$

66. $(a^2 + b^2)^2 - 4a^2b^2$

67. $4y^4 - 5y^2 - 9$

68. $a^6 - 2a^3b^3 + b^6$

69. $4c^4 - 7c^3 - 2c^2$

[Supplementary exercises are in Part N on page 4-129.]

B. Simplify.

Sample. $\sqrt{4x^6}$

Solution. $\sqrt{4x^6} = \sqrt{2^2 \cdot (x^3)^2}$

$= \sqrt{(2x^3)^2}$

$= |2x^3|$.

1. $\sqrt{x^2y^6}$

2. $\sqrt{y^4 - 4y^2 + 4}$

3. $\sqrt{a^6 + 2a^4 + a^2}$

4. $\sqrt{9 - 6u^8 + u^{16}}$

5. $\sqrt{25x^{12}y^8}$

6. $\sqrt{(x^4 + y^4)^2 - (x^4 - y^4)^2}$

7. $\sqrt{(a + b)^2(x + 2y)^4}$

8. $\sqrt{(x^2 + y^2)^2(x^2 - y^2)^4}$

9. $\sqrt{\frac{x^6y^4}{z^2}}$

10. $\sqrt{\frac{a^4 - 2a^2 + 1}{a^4 + 2a^2 + 1}}$

* * *

Given two or more pronumeral expressions, a common factor which is a multiple of each common factor of the given expressions is called a highest common factor of the expressions. For example, both 'xy' and 'yx' are highest common factors of '2xy' and '3xy'.

* * *

C. Find an HCF of the given pronumeral expressions.

Sample 1. x^5y^4, x^3y^2, x^2y^6

Solution. An HCF of 'x^5y^4', 'x^3y^2', and 'x^2y^6' is 'x^2y^2'. [Another is 'y^2x^2'.]

60. $x^2(x + 4)(x + 2)$

61. $(x - 2)(x + 2)(x - 1)(x + 1)$

62. $(y - 3)(y + 3)(y^2 + 1)$

63. $xy^2(3x + 4)(2x - 1)$

64. $(u^3 - 3y^3)(u^3 + 3y^3)$

65. $[(x + y) - (x - y)][(x + y) + (x - y)]$, [or: $4yx$]

66. $(a - b)(a - b)(a + b)(a + b)$

67. $(2y - 3)(2y + 3)(y^2 + 1)$

68. $(a^3 - b^3)^2$ [You may want to introduce the standard factorization for '$a^3 - b^3$'.]

69. $c^2(4c + 1)(c - 2)$

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Answers for Part B.

1. $|xy^3|$

2. $|y^2 - 2|$

3. $|a(a^2 + 1)|$ [or: $|a|(a^2 + 1)$]

4. $|3 - u^8|$

5. $5x^6y^4$

6. $2x^2y^2$

7. $|(a + b)(x + 2y)^2|$ [or: $|(a + b)|\ (x + 2y)^2$]

8. $(x^2 + y^2)(x^2 - y^2)^2$

9. $\left|\frac{x^3y^2}{z}\right|$, $[z \neq 0]$

10. $\left|\frac{a^2 - 1}{a^2 + 1}\right|$ [or: $\frac{|a^2 - 1|}{a^2 + 1}$]

*

Answers for Part C [on pages 4-80 and 4-81].

[As indicated in the Samples, there is more than one HCF for the pronumeral expressions given in each exercise. We give just one.]

1. $2yz$
2. $5s$
3. a^2bc
4. x^2y^9
5. $3xy$
6. $4m^2p^3$
7. $(x - 2y)(x + y)(x - y)$
8. $a - 2$
9. $x(x + 1)$
10. $2x - 3$
11. 1
12. $n - 2r$
13. $5n(n - 3)$
14. $cd(c + 2d)$
15. $5(s - 5)$
16. $(a + b)(2a + b)$

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Answers for Part D [on page 4-82].

1. $7xy(x + 3y)$
2. $ab(3ac^4 - 5)$
3. $3xyz(2xyz + 1)$
4. $ab^3c(9a^4c - 2)$
5. $13abc(1 + 2ab^2c^3 - 5ab)$
6. $2(6x^2y^2 + 2y^2z^2 + z^2x^2)$
7. $5(x + y)^2(x - y)^3(2 + x + y)$
8. $(2a - b)(a + 3b)\ [3 + 7(2a - b)(a + 3b)]$

Sample 2. $8a^3b^2c^4$, $12ab^3c^2$

Solution. An HCF is '$4ab^2c^2$'.

1. $10xy^2z$, $6y^3z$, $2yz^2$

2. $15r^3s^2$, $30rs^3$, 10^2s

3. $6a^2bc^3$, $5a^3b^2c$

4. x^5y^{12}, $3x^4y^9$, $9x^2y^{23}$

Sample 3. $6a + 18b$, $12a + 30b$

Solution. For each a and b,

$$6a + 18b = 6(a + 3b)$$

and $$12a + 30b = 6(2a + 5b).$$

So, an HCF is '6'.

Sample 4. $x^2 - y^2$, $x^2 + 4xy + 3y^2$

Solution. For each x and y,

$$x^2 - y^2 = (x - y)(x + y)$$

and $$x^2 + 4xy + 3y^2 = (x + 3y)(x + y).$$

So, an HCF is '$x + y$'.

5. $6x^2y + 15xy^2$, $21x^2y - 6xy^2$

6. $12m^3p^5 + 20m^2p^3r^2$, $8m^5p^4 + 24m^2p^5r$

7. $(x - 2y)^2(x + y)(x - y)^3$, $(x - 2y)(x + y)^2(x - y)$

8. $a^2 - 5a + 6$, $a^2 - 4a + 4$

9. $x^3 + 3x^2 + 2x$, $x^3 - x$

10. $12x^2 - 4x - 21$, $2x^2 + 7x - 15$

11. $x^3y - 5x^2y^2 - 14xy^3$, $x^2 - 3xy + 2y^2$

12. $n^2 + nr - 6r^2$, $n^2 - 7nr + 10r^2$

13. $5n^3 - 45n$, $10n^3 - 50n^2 + 60n$

14. $c^3d + 3c^2d^2 + 2cd^3$, $c^3d + c^2d^2 - 2cd^3$

15. $5s^2 - 125$, $5s^3 - 50s^2 + 125s$

16. $(a + b)(2a^2 - ab - b^2)$, $(2a^2 + 3ab + b^2)$

D. The idea of the HCF is useful in factoring pronumeral expressions.

Sample. Factor:

$$15x^2y^3z^2 + 20x^3y^2z^5 + 35x^2y^3z.$$

Solution. Since an HCF of the three addends is '$5x^2y^2z$', it follows that a factorization of the given expression is:

$$5x^2y^2z(3yz + 4xz^4 + 7y).$$

Factor these expressions using the idea of the HCF.

1. $7x^2y + 21xy^2$
2. $3a^2bc^4 - 5ab$
3. $6x^2y^2z^2 + 3xyz$
4. $9a^5b^3c^2 - 2ab^3c$
5. $13abc + 26a^2b^3c^4 - 65a^2b^2c$
6. $12x^2y^2 + 4y^2z^2 + 2z^2x^2$
7. $10(x + y)^2(x - y)^3 + 5(x + y)^3(x - y)^3$
8. $3(2a - b)(a + 3b) + 7(4a^2 - 4ab + b^2)(a^2 + 6ab + 9b^2)$

[Supplementary exercises are in Part O on page 4-129.]

* * *

Given two or more pronumeral expressions, a common multiple which is a factor of each common multiple of the given expressions is called a lowest common multiple of the expressions. For example, both '$6xy$' and '$6yx$' are lowest common multiples of '$3xy$' and '$2x$'.

* * *

E. Find an LCM of the given pronumeral expressions.

Sample 1. a, $2b$, $7c$

Solution. An LCM is '$14abc$'.

Sample 2. $5x^3y^5$, $20x^2y^8$

Solution. An LCM of '$5x^3y^5$' and '$20x^2y^8$' is '$20x^3y^8$'. [Another is '$20y^8x^3$'.]

[4-83]

D. ...

Answers for Part E [which begins on page 4-82].

[Again, we give just one expression for each exercise, though there are others which are LCMs of the given pronumeral expressions.]

1. $21xyz$
2. $20xyz$
3. y^5
4. $12x^4$
5. $10a^2b^2$
6. $42xy$
7. $12a^3b^3c^4$
8. $20x^2y^2z^2$
9. $18u^3v^4$
10. $12a^2b^2x^2y^2$
11. $3(a + b)$
12. $4(x - y)$
13. $6(x - 1)$
14. $15(a + 5b - c)$
15. $(a + b)(a - b)$
16. $3(x + y)^2$
17. $10(x - y)$
18. $x - 2y$
19. $(2x - y)(x - 2y)$
20. $a^2 - b^2$
21. $(x + 1)(2x - 1)$
22. $a^4 - b^4$
23. $(x + 1)(x - 3)(x - 1)(x + 2)$
24. $(3s - 10)(s + 1)(10s + 3)$
25. $(x - y)^2(x + 2y)^2(x + y)^2(2x + y)$

*

Answers for Part F [on pages 4-84, 4-85, and 4-86].

1. $\frac{23}{24}$
2. $\frac{1}{6}$
3. $-\frac{13}{48}$
4. $\frac{29}{72}$
5. $-\frac{11}{98}$
6. 1
7. $\frac{4x^3y + 15}{6x^2y^2}$
8. $\frac{5x - 6x^2 + 3}{x^3}$
9. $-\frac{3}{4x^2}$
10. $\frac{6x^2 + xy - 4y^2}{12x^2y^2}$
11. $-\frac{7x}{4(x + 3)}$
12. $\frac{6y - 7}{(y - 5)(y - 1)}$
13. $\frac{2x - 5}{x - 1}$
14. $\frac{4x}{(x + 2)(x + 1)(x - 1)}$
15. $\frac{y - x}{xy}$
16. $\frac{7u^2 + 8u + 8}{(3u + 2)(3u - 2)(u + 2)}$
17. $\frac{8a^4 + 14a^3 + 8a^2 - 16a}{(a^2 + 4)(a + 2)(a - 2)(a + 3)}$
18. $-\frac{y(5y + 6)}{3(y - 3)(y + 3)}$

1. $x, \ 3y, \ 7z$
2. $4x, \ 5xy, \ 5yz$
3. $y, \ y^2, \ y^5$
4. $2x, \ 3x^2, \ 4x^4$
5. $2a^2b, \ 10b^2a$
6. $7, \ 14xy, \ 21y$
7. $2a^3b^2c, \ 4abc^2, \ 6a^2b^3c^4$
8. $x^2y, \ 4xyz, \ 10y^2z^2$
9. $9u^3v^2, \ 2uv^4, \ 6u^2v$
10. $3a^2x^2, \ 2abxy, \ 4b^2y^2$

<u>Sample 3</u>. $7, \ 7x + 4$

<u>Solution</u>. An LCM of '7' and '$7x + 4$' is '$7(7x + 4)$'. [Another is '$49x + 28$'.]

<u>Sample 4</u>. $x^2 - y^2, \ x^2 - 2xy + y^2$

<u>Solution</u>. For each x and y,

$$x^2 - y^2 = (x - y)(x + y)$$

and $$x^2 - 2xy + y^2 = (x - y)^2.$$

So, an LCM of '$x^2 - y^2$' and '$x^2 - 2xy + y^2$' is

'$(x - y)^2(x + y)$'.

11. $3, \ a + b$
12. $4, \ 2(x - y)$
13. $6x - 6, \ 3$
14. $a + 5b - c, \ 15$
15. $a + b, \ a - b$
16. $3(x + y), \ (x + y)^2$
17. $5(x - y), \ 10(x - y)$
18. $x - 2y, \ x - 2y$
19. $2x - y, \ x - 2y$
20. $a - b, \ a^2 - b^2$
21. $x + 1, \ 2x - 1$
22. $a^4 - b^4, \ a^2 - b^2, \ a + b$
23. $x^2 - 2x - 3, \ x^2 - 1, \ x + 2$
24. $3s^2 - 7s - 10, \ 30s^2 - 91s - 30$
25. $(x - y)(x + 2y)^2(x + y), \ (2x + y)(x - y)^2, \ (x + y)^2$

[Supplementary exercises are in Part P on page 4-130.]

F. Simplify.

Sample 1. $\frac{5}{72} + \frac{17}{48}$

Solution. Since $72 = 3 \cdot 24$ and $48 = 2 \cdot 24$, the LCM of 72 and 48 is $6 \cdot 24$, or 144. $144 = 72 \cdot 2$, and $144 = 48 \cdot 3$.

$$\begin{aligned} \frac{5}{72} + \frac{17}{48} &= \frac{5}{72} \cdot \frac{2}{2} + \frac{17}{48} \cdot \frac{3}{3} \\ &= \frac{10}{144} + \frac{51}{144} \\ &= \frac{61}{144}. \end{aligned}$$

1. $\frac{3}{8} + \frac{7}{12}$
2. $\frac{7}{15} - \frac{3}{10}$
3. $\frac{5}{24} + \frac{7}{12} - \frac{17}{16}$
4. $\frac{7}{36} + \frac{5}{24}$
5. $\frac{5}{14} - \frac{23}{49}$
6. $\frac{19}{21} - \frac{6}{35} + \frac{4}{15}$

Sample 2. $\frac{3}{5x^3y^5} + \frac{2}{7x^2y^8}$

Solution. An LCM of '$5x^3y^5$' and '$7x^2y^8$' is '$35x^3y^8$'.

For each x, for each y,

$35x^3y^8 = 5x^3y^5 \cdot 7y^3$ and $35x^3y^8 = 7x^2y^8 \cdot 5x$.

$$\begin{aligned} \frac{3}{5x^3y^5} + \frac{2}{7x^2y^8} &= \frac{3}{5x^3y^5} \cdot \frac{7y^3}{7y^3} + \frac{2}{7x^2y^8} \cdot \frac{5x}{5x} \\ &= \frac{21y^3}{35x^3y^8} + \frac{10x}{35x^3y^8} \\ &= \frac{21y^3 + 10x}{35x^3y^8}. \end{aligned}$$

[Note: We assume in this sample and in the exercises which follow that values of the pronumerals which would convert a denominator into a name for 0 have been excluded from the domains of the pronumerals.]

7. $\dfrac{2x}{3y} + \dfrac{5}{2x^2y^2}$

8. $\dfrac{5}{x^2} - \dfrac{6}{x} + \dfrac{3}{x^3}$

9. $\dfrac{2x^2 + 4x - 3}{4x^2} - \dfrac{x + 2}{2x}$

10. $\dfrac{2x + y}{4xy^2} - \dfrac{x + 2y}{6x^2y}$

Sample 3. $\dfrac{x - y}{x^2 + 2xy + y^2} - \dfrac{x}{x^2 - y^2}$

Solution. $\dfrac{x - y}{x^2 + 2xy + y^2} - \dfrac{x}{x^2 - y^2}$

$$= \frac{x - y}{(x + y)^2} - \frac{x}{(x - y)(x + y)}$$

$$= \frac{x - y}{(x + y)^2} \cdot \frac{x - y}{x - y} - \frac{x}{(x - y)(x + y)} \cdot \frac{x + y}{x + y}$$

$$= \frac{(x - y)^2}{(x + y)^2(x - y)} - \frac{x(x + y)}{(x + y)^2(x - y)}$$

$$= \frac{(x^2 - 2xy + y^2) - (x^2 + xy)}{(x + y)^2(x - y)}$$

$$= \frac{-3xy + y^2}{(x + y)^2(x - y)} \quad \left[\text{or: } \frac{y(y - 3x)}{(x + y)^2(x - y)}\right].$$

11. $\dfrac{x}{4x + 12} - \dfrac{2x}{x + 3}$

12. $\dfrac{4y + 3}{y^2 - 6y + 5} + \dfrac{2}{y - 1}$

13. $\dfrac{x + 2}{x + 1} + \dfrac{x - 3}{x - 1} - \dfrac{2x}{x^2 - 1}$

14. $\dfrac{x + 1}{x^2 + x - 2} - \dfrac{x - 1}{x^2 + 3x + 2}$

15. $\dfrac{y}{x^2 + xy} - \dfrac{x}{xy + y^2}$

16. $\dfrac{u + 5}{9u^2 - 4} + \dfrac{2u - 1}{3u^2 + 4u - 4}$

17. $\dfrac{6a^3}{a^4 - 16} + \dfrac{2a}{a^2 + 5a + 6}$

18. $\dfrac{y}{3y - 9} + \dfrac{2y}{2y + 6} - \dfrac{3y^2}{y^2 - 9}$

(continued on next page)

<u>Sample</u> <u>4</u>. $\frac{5}{x - y} - \frac{3y - 2x}{y - x}$

<u>Solution.</u> $\frac{5}{x - y} - \frac{3y - 2x}{y - x}$

$$\left.\begin{aligned} &= \frac{5}{x - y} - \frac{3y - 2x}{-(x - y)} \\ &= \frac{5}{x - y} - \frac{-(3y - 2x)}{x - y} \end{aligned}\right\} \quad \underline{[\text{Explain}]}$$

$$= \frac{5 - -(3y - 2x)}{x - y}$$

$$= \frac{5 + 3y - 2x}{x - y}.$$

19. $\frac{7a}{a - b} - \frac{10 + b}{b - a}$

20. $\frac{2c - 3}{d - 2c} + \frac{-5 + c}{2c - d}$

21. $\frac{3s + 1}{s^2 - 9} - \frac{-2s - 3}{3 - s}$

22. $\frac{-3a + b}{3a - b - 2c} + \frac{c - 2b}{b + 2c - 3a}$

23. $\frac{2r - s}{r^2 - s^2} + \frac{4}{4s - 4r}$

24. $\frac{c - d}{d - c} - \frac{2a - 3b}{3b - 2a}$

25. $\frac{-5c + d}{2c - d} + \frac{2d - 3c}{d - 2c}$

26. $\frac{3n}{n + 2} - \frac{n}{10 + 5n}$

27. $\frac{2d - 5}{1 - 2d + d^2} + \frac{3}{d - 1}$

28. $\frac{n}{n^2 - nk} - \frac{k}{nk - k^2}$

29. $\frac{h + 1}{h + 2} - \frac{h + 2}{2 - h} - \frac{2h}{h^2 - 4}$

30. $\frac{-4t^2}{16 - t^2} + \frac{t}{4t - 16} - \frac{2t}{2t + 8}$

31. $-\frac{5n - 4t}{4t - 5n} + \frac{r - s}{s - r}$

32. $\frac{2r + 1}{4r^2 - 4r + 1} - \frac{1 - 2r}{1 - 4r^2}$

33. $\frac{3a + 2c}{bc^2 - ba^2} - \frac{5}{ba - bc}$

34. $\frac{147a}{a - b} + \frac{b + 146a}{b - a}$

☆35. $\frac{2 - 3n}{9 - 8n - n^2} - \frac{5n + 6}{n^2 + 29n - 30}$

[Supplementary exercises are in Part Q, pages 4-130 through 4-131.]

19. $\frac{7a + 10 + b}{a - b}$ 20. $\frac{c + 2}{d - 2c}$ 21. $\frac{-2(s^2 + 3s + 4)}{(s + 3)(s - 3)}$

22. $\frac{-3a + 3b - c}{3a - b - 2c}$ 23. $\frac{r - 2s}{(r + s)(r - s)}$ 24. 0

25. $\frac{2c + d}{d - 2c}$ 26. $\frac{14n}{5(n + 2)}$ 27. $\frac{5d - 8}{(d - 1)(d - 1)}$

28. 0 29. $\frac{2h^2 + h + 2}{(h + 2)(h - 2)}$ 30. $\frac{13t^2 + 20t}{4(t - 4)(t + 4)}$

31. 0 32. $\frac{8r}{(2r - 1)(2r - 1)(2r + 1)}$ 33. $\frac{8a + 7c}{b(c - a)(c + a)}$

34. 1 ☆35. $-\frac{2n^2 - 37n + 114}{(n + 9)(n - 1)(n + 30)}$

*

Quiz.

In the following expressions, substitute '3' for 'c', '−2' for 'd', '5' for 'r', and '−7' for 's', and simplify.

1. $\frac{c^2 - d^2}{c - d}$ 2. $\frac{2r - 6s}{8c^2 + 16cd + 8d^2}$ 3. $\frac{rs^2}{r^2s}$

4. $\frac{(r + s)^2}{(c + d)^2}$ 5. $\frac{rc - rd}{c - d}$ 6. $\frac{(1/5)r + (1/5)s}{(1/5)c - (1/5)d}$

7. $\frac{(cd)^2}{(rd^2)^2}$ 8. $\frac{rc^2 + 2cdr - 3d^2r}{c^2s - d^2s}$

Solve and check.

9. $n = .5(90 - n)$ 10. $a^2 - 11a = 60$ 11. $3b^2 = 147$

12. $-5/3 = 3t - 6t^2$ 13. $3.5g^2 - 31.5 = 0$ 14. $10p = 39 - p^2$

*

Answers for Quiz.

1. 1 2. 13/2 3. −7/5 4. 4 5. 5

6. −2/5 7. 9/100 8. 15/7 9. 30 10. 15, −4

11. 7, −7 12. 5/6, −1/3 13. 3, −3 14. 3, −13

[4-86]

[4-87]

sect in a set consisting of a single point, at least one of whose components is not an integer. It is easy to pick the point first, and then find the equations. For example, suppose the point is (3, 1/2). Then the equations:

$$x + 2y = 4, \; x + 4y = 5$$

have noncrossing graphs on a number plane lattice picture but intersecting, straight line graphs on a number plane picture.

6. [It is redundant to say 'two different equations' since two equations must be different [except for the case where the equations are copies of each other]. But, this redundancy serves the purpose of emphasis.] Any two equivalent equations will do.

7. $\{(x, y): x + y = y + x\}$; $\{(x, y)$ x and y integers: $x + y = y + x\}$.

   [The set selector in the first name could be a true sentence such as '2 = 2' or '7 > 3'.]

8. Any sentence whose corresponding universal generalization is false will do. For example, '$x + y = x + y + 1$'. Also, any false sentence will do. For example, '1 = 2'. [Ask the class if it is the case that a sentence whose number plane locus is the empty set is also a sentence whose number plane lattice locus is the empty set. Then ask if a sentence whose number plane lattice locus is the empty set is also a sentence whose number plane locus is the empty set. For a given sentence, its number plane lattice locus is a subset of its number plane locus. Hence, the answer to the first question is 'yes' since a subset of $\emptyset$ is $\emptyset$. But, the answer to the second question is 'no' because it is possible to have a set of ordered pairs of real numbers none of whose subsets contains an ordered pair of integers. For example, $\{(x, y): x + y = 1/2\}$.]

9. (a) $x + 5y = 16$ (b) $3y - 7x = 9$

   (c) There is no value of 't' which will do the job. Students should conclude that the solution set of each equation of the form:

   $$3tx - 5y = -18$$

   intersects the y-axis at (0, 18/5) and at no other point.

   (d) $2x^2 + 11(3/4)y = 13(3/4)$ (e) No value of 't' will do the job.

[4-87]

3. Equations whose straight line graphs contain the graph of the origin are equivalent to equations of the form:

$$Ax + By = 0,$$

where not both A and B are 0. As in Exercise 2, there are many equations whose graphs are not straight lines and which contain the graph of the origin. An interesting class exercise is to write equations on the board and ask the class to state whether or not the graph of each contains the graph of the origin. Here are several such equations.

$y = 7x$...yes	$y = 7x + 9$...no
$y^2 + x^2 = 0$...yes	$y^2 - 2x^2 = 0$...yes
$x^2y^3 = 3x + 5y$...yes	$x^3y^2 = 3y + 5x - 7$...no
$9x - 3y = 0$...yes	$7x + 7y + 5 = 0$...no
$(x - 5)^2 + y^2 = 25$...yes	$(x - 3)^2 + (y + 4)^2 = 25$...yes
$y^2 = \dfrac{2x}{x - y}$...no	$y^2(x - y) = 2x$...yes

4. Students will probably give answers like:

$$x = 3,\ x = 4 \quad \text{or:} \quad y = 7,\ y = 9.$$

Others who like to search for unusual answers might come up with:

$$x = x + 1,\ x = x + 2 \quad \text{or:} \quad xy = yx,\ x + y = y + x + 1$$

or even with:

$$x^2 + y^2 = 1,\ x^2 + y^2 = 2.$$

In class you might ask for an equation whose locus intersects the locus of '$y = 2x$' in the empty set. If no one suggests an equation of the form '$y = 2x + A$', propose '$y = 2x + 1$', and ask if someone can explain why '$y = 2x + 1$' and '$y = 2x$' have loci which intersect in the empty set. A good explanation might be that if the loci did intersect in some point, the second component of that point would be equal to 1 more than itself, which is impossible. Students should then be able to produce lots of equations whose loci are parallel to the locus of '$y = 2x$'. Repeat this procedure with '$y = 3x$', then with '$y = 5x + 7$', then with '$y = -2x + 3$', then with '$y + 7x - 5 = 0$', and finally with '$4y + 3x - 2 = 0$'. [Keep this work informal.]

5. The clever student will answer this question quickly by giving two equations each of whose number plane lattice loci is the empty set and whose number plane loci are a vertical and a horizontal line. For example: $x = 1/2$, $y = 1/2$. The student needs to realize that his job here is to find two equations whose number plane loci inter-

[4-87]

Answers for MISCELLANEOUS EXERCISES.

[Parts A through E provide an interesting review of Unit 4. We have tried to include exercises which are different from those in the body of the text but which call for applications of the principles learned in the unit.]

A. 1. (a) a straight line [that is, a picture of a number plane straight line]

[This exercise and Exercise 1(b) point up the ambiguity in a command such as:

Draw the graph of '$x = 5$'.

The brace-notation is much to be preferred:

Draw a picture of $\{x: x = 5\}$,

or: Draw a picture of $\{(x, y): x = 5\}$.]

(b) The graph of '$y > -1$' on a number line picture is a half-line. The graph of '$y > 1$' on a number plane picture is a half-plane.

2. A clever student might write: $x = -2$, $y = 5$. [These are equations in 'x' and 'y' even though each equation contains only one of these pronumerals. An equation in 'x' and 'y' might be thought of as one which could serve as a set selector in the brace-notation name for a set of ordered pairs where the index is '(x, y)'. To stop arguments on this point, you may suggest the equations:

$$x = -2 + 0y, \; y = 5 + 0x,$$

instead of:

$$x = -2, \; y = 5.$$

Of course, these are different equations but the graphs are the same.]

Other equations whose straight line graphs contain the graph of $(-2, 5)$ are equivalent to equations of the form:

$$A(x + 2) + B(y - 5) = 0,$$

where not both A and B are 0. Of course, there are many equations whose graphs are not straight lines and which contain the graph of $(-2, 5)$. For example:

$$x^2 + y^3 = 129.$$

MISCELLANEOUS EXERCISES

A. 1. (a) In Unit 3 you learned that the graph of '$x = 5$' [on a picture of the number line] consists of a single point. What is the graph of '$x = 5$' on a number plane picture?

(b) What is the graph of '$y > -1$' on a number line picture? What is the graph of '$y > 1$' on a picture of the (x, y)-plane?

2. Write two equations [in 'x' and 'y'] whose graphs on a number plane picture cross at the graph of (−2, 5).

3. Write three equations [in 'x' and 'y'] whose graphs on a picture of the number plane have the graph of the origin as their common point.

4. Write two equations whose loci in (x, y) intersect in the empty set.

5. Write two equations whose loci in the number plane lattice intersect in the empty set, but whose loci in the number plane intersect in a set consisting of a single point.

6. Write two different equations which have the same loci in the number plane.

7. Use brace-notation [{(x, y): . . . }] to name a set whose elements [members] are all the points of the number plane. Of the number plane lattice.

8. Write an equation whose solution set in (x, y) is the empty set.

9. In each of the following sentences, replace the 't' by a numeral such that the locus of the resulting sentence will include the ordered pair listed.

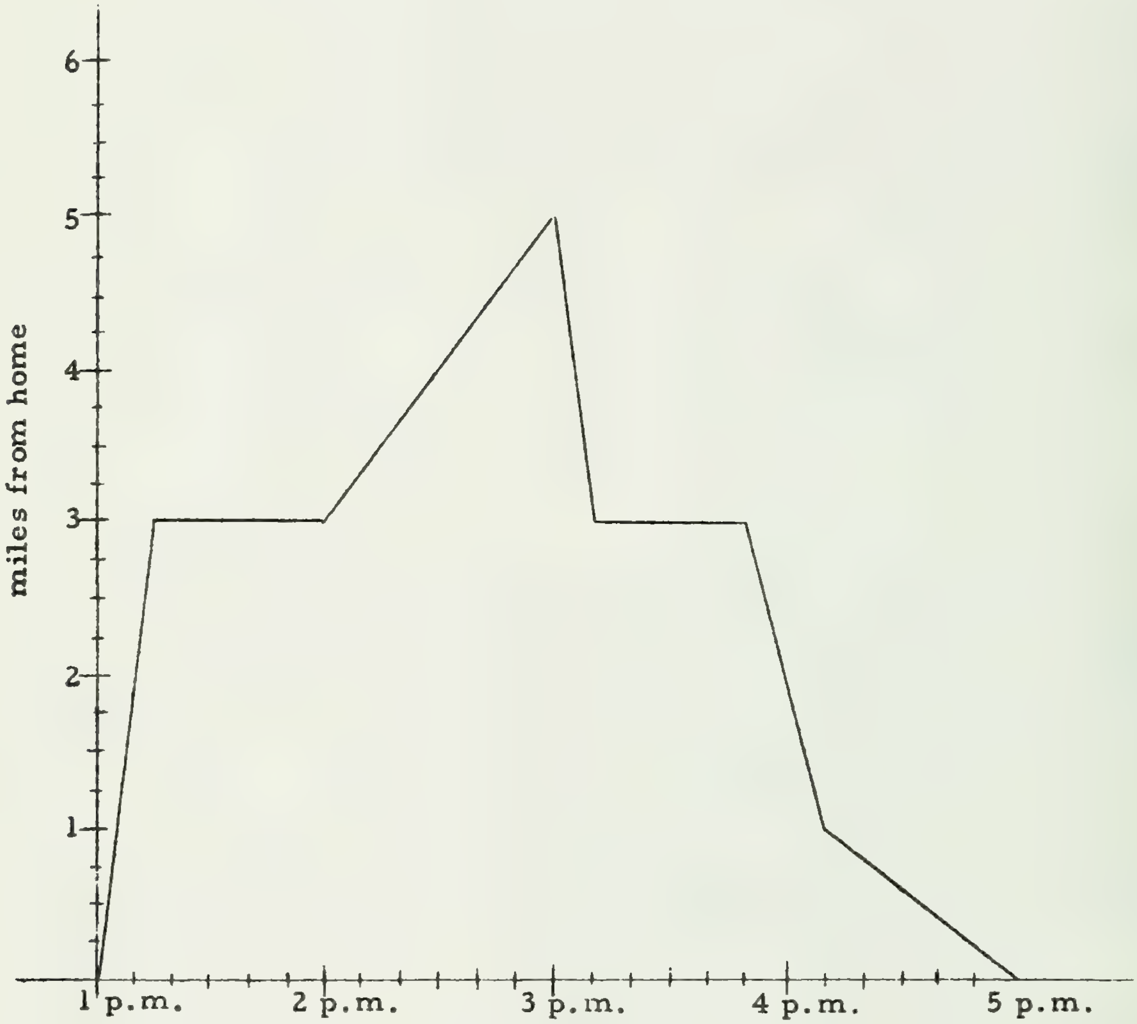
(a) $x + ty = 16$; (6, 2) (b) $3y - 7x = t$; (−3, −4)

(c) $3tx - 5y = -18$; (0, −7) (d) $2x^2 + 11ty = 13t$; (3, −1)

(e) $5tx - 6y + 7 = 0$; (0, 0)

(continued on next page)

10. Here is a graph which is an approximate record of a bicycle trip Ed took one day last summer. He started from home at 1:00 p.m. and kept track of his distance from home at various times. [For example, at 3:10 p.m. he was 3 miles from home.]



(a) How far from home was he at 2:30 p.m.?

(b) At what time was he 2 miles from home?

(c) By what time had he traveled a total of 8 miles?

(d) What was his average speed during the first 15 minutes?

(e) Did he stop during the trip?

(f) What was his average speed during the period from 3:00 p.m. to 3:10 p.m.?

10. [Be sure students understand how a graph like this is made. Some students think that the graph is a picture of the rocky road over which Ed traveled, the flat places being level road! Pretend that Ed has an odometer on his bike and that when he started at 1 p.m., he set his odometer at '0'. Then, from 1 p.m. to 3 p.m., he made pairs of readings at several points along the way. Perhaps every 10 minutes he read both his watch and his odometer. At 3 p.m. when he decided to return home on the same road [or on another road which would bring him home after traveling 5 miles], he reset his odometer at '0'. At 3:10 p.m. when his odometer read '2', he knew he was 3 miles from home. [Maybe, instead of resetting the odometer, he fixed the gears in it so that the dials would move backward.] The details of the story you tell or that the students invent are unimportant. The important thing is to invest this problem with life so that the student can imagine himself taking the trip and can see the connection between the trip and the graph. The questions following the chart are designed to test the student's ability to interpret the chart. It will help if he has some feeling for how it was made.]

(a) 4 miles (b) 1:10 <u>and</u> 4:00 (c) 4:00

(d) At 1:15 he had traveled 3 miles. So, it took him 15 minutes to travel 3 miles. Hence, his average speed was 12 miles per hour.

(e) He stopped for the period 1:15 - 2:00, and for the period 3:10 - 3:50. [He may have stopped at 3:00 to turn his bicycle around and start for home. If so, the chart shows that he accomplished this in no time at all. On the other hand, he may have made a U-turn at 3:00 and not have stopped at all, a conclusion which is not contradicted by the chart.]

(f) 12 miles per hour

(g) Since he traveled 2 miles during this period of 50 minutes his average speed was 2.4 miles per hour.

(h) 3:50 - 4:10 [Students can answer this question by computing. But, they should see that the steeper portion of the graph indicates the greater speed.]

(i) 10 miles (j) 2.5 m.p.h.

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position which is still on the vertical line. Or, the transformation can be thought of as one which slides all vertical lines along themselves. [For the number line illustration, you can visualize dividing by 2 as pushing dots toward the graph of 0 but never getting past it. Then it is easy to see that $\{x: 0 \leq x \leq 1\}$ is closed under this operation. So are $\{x: -1 \leq x \leq 0\}$ and $\{x: 0 \leq x \leq 30\}$, for example.] Once the student has convinced himself that the subset is closed, the further question about staying in the set if you make additional moves is superfluous.

In connection with the given illustration you should ask if $\{(x, y): y = 2\}$ has this same property as $\{(x, y): x = 7\}$. [You can use the expression 'closed under this moving rule' if you wish.] The answer is 'no' since the horizontal line moves from its original position to one which is 3 units higher. To see this, give a starting point in $\{(x, y): y = 2\}$, say (9, 2), and show that the moving rule takes you to (9, 5), a point which is not in $\{(x, y): y = 2\}$.

Further questions you can ask about the illustration are:

(1) Name some other sets which are closed under this moving rule. [$\{(x, y): x = -5\}$, $\{(x, y): x = 0\}$, etc.]

(2) Is $\{(x, y): x = 5 \text{ and } y \geq 0\}$ closed under this moving rule? [Yes.]

(3) Is $\{(x, y): x = 5 \text{ and } y \leq 0\}$ closed under this moving rule? [No.]

(4) Is the number plane itself closed under this moving rule? [Yes.]

(5) Is the empty set closed under this moving rule? [Yes.]

*

Here are answers for Exercise 11.

(a) This transformation may be visualized by thinking of it as shifting dots horizontally to the left. Since the graphs of (2) and (3) are horizontal lines, sets (2) and (3) are closed under this rule. [Sets (1) and (4) are not closed. Try some points.]

(b) This transformation may be visualized by thinking of it as making dots change places with their mirror images, the edge of the mirror being placed along the '$y = x$'-line. Hence, sets which are closed under this rule are just those whose graphs are symmetric with respect to the '$y = x$'-line. Such sets are (1) and (3) among those listed. In answering this question, especially in determining that (3) is closed, students will probably have discovered that the set in question is closed under this rule if and only if interchanging 'x' and 'y' in the set selector gives you an equivalent selector. [An easy test point for (4) is $(\sqrt{5}, 0)$.]

rule:

$$x \rightarrow x + 7$$

expresses the operation adding 7.] Thus, the first mentioned moving rule:

$$(x, y) \rightarrow (x, y + 3)$$

expresses the transformation which is the set of ordered pairs of number plane points some of which are

$$((2, 7), (2, 10)) \ ((1, 8), (1, 11)), \text{ and } ((-2, 5), (-2, 8)).$$

In order to visualize a transformation, it is helpful to give a geometric interpretation. For example, we can visualize the operation adding 7 by thinking of each dot on a number line picture as jumping 7 units to the right. Similarly, we can visualize the transformation expressed by:

$$(x, y) \rightarrow (x, y + 3)$$

by thinking of each dot in a number plane picture as jumping 3 units straight up in the picture.

Now, for certain transformations on the number plane [or on the number line], there are certain subsets of the number plane [or of the number line] which have an interesting property. Consider the transformation on the number line which is the operation dividing by 2. The moving rule is:

$$x \rightarrow \frac{x}{2}.$$

Now, consider the subset $\{x: 0 \leq x \leq 1\}$, that is, the set of all numbers from 0 to 1. Under the operation dividing by 2, each member of this subset is paired with a member of the same subset. Geometrically, each dot in a graph of this subset jumps to another dot in the same graph--no dot can jump "out" of the graph. On the other hand, some of the dots in the graph of $\{x: 1 \leq x \leq 3\}$ do jump out of the graph of this subset under the operation dividing by 2. We say that $\{x: 0 \leq x \leq 1\}$ is closed under the operation dividing by 2, and that $\{x: 1 \leq x \leq 3\}$ is not closed under this operation.

The notion of closure of a subset under a transformation can be carried over to the number plane, and it is just this idea that Exercise 11 seeks to develop. The subset, $\{(x, y): x = 7\}$, of the number plane is closed under the transformation expressed by:

$$(x, y) \rightarrow (x, y + 3)$$

This is an easy thing to see geometrically, and we want the students to develop this geometric intuition. The transformation can be thought of as making dots jump straight up in the picture. So, since the graph of the set in question is a vertical line, each dot on it jumps to another

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(k) Again, this question can be answered by computing, but students should see how to answer it "graphically". Just extend the picture of the interval with end points (3:50, 3) and (4:10, 1) until it crosses the horizontal axis. It does so at the graph of (4:20, 0). So, he would have arrived home at 4:20 p.m. .

(l) This may sound like hair-splitting, but the question does get at the student's understanding of the graph. Sharp corners mean abrupt changes in velocity. For example, Ed didn't attain a speed of 12 miles per hour instantaneously at the start of the trip. The graph should look something like Figure 1 at the 1:00-point.

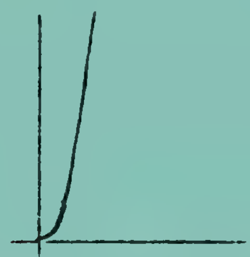

Fig. 1

Also, at 1:15 his velocity didn't change instantaneously from 12 miles per hour to 0 miles per hour [unless he smashed into a brick wall!]. It is more likely that the graph would look like Figure 2 at that point.

Fig. 2

*

Exercise 11 deals with properties of certain transformations on the number plane. A transformation on the number plane is analogous to an operation on a set of numbers. A number plane transformation is a set of ordered pairs of number plane points. We describe a transformation by talking about a moving rule for a number plane game. [We could describe an operation on the set of real numbers by talking about a moving rule for a number line game. For example, the moving

(g) What was his average speed during the period from 3:00 p.m. to 3:50 p.m.?

(h) During which period was he traveling faster--from 3:50 p.m. to 4:10 p.m., or from 4:10 p.m. to 5:00 p.m.?

(i) How many miles did he travel altogether?

(j) What was his average speed for the entire trip?

(k) If, from 4:10 p.m. until arrival at home, Ed had traveled at the same average speed as that maintained from 3:50 p.m. to 4:10 p.m., at what time would he have arrived at home?

(l) Notice the "corners" at various points of the graph. If the chart were drawn accurately, is it likely that there would be corners like these? Explain.

11. Suppose you are playing a number plane game in which the "moving" rule is:

$$(x, y) \rightarrow (x, y + 3).$$

Then, if you start at a point in $\{(x, y): x = 7\}$, say, $(7, 2)$, the first move takes you to $(7, 5)$, the second move takes you to $(7, 8)$, and, in fact, no matter how many moves you make, the point you reach is in $\{(x, y): x = 7\}$.

In each of the following exercises, you are given a moving rule and several sets. Tell for which sets it is the case that, if you start from a point in the set, you can't get out of the set no matter how many moves you make.

(a) $(x, y) \rightarrow (x - 3, y)$

(1) $\{(x, y): x + 3 = y\}$ (2) $\{(x, y): y = 2\}$

(3) $\{(x, y): x + 2y = 10 + x\}$ (4) $\{(x, y): x = 3\}$

(b) $(x, y) \rightarrow (y, x)$

(1) $\{(x, y): x + y = 9\}$ (2) $\{(x, y): x - y = 9\}$

(3) $\{(x, y): x + 2xy + y = 4\}$ (4) $\{(x, y): x^2 + 3y^2 = 5\}$

(continued on next page)

(c) $(x, y) \rightarrow (x, -y)$

(1) $\{(x, y): x + 7 = |y|\}$ (2) $\{(x, y): |x| - y = 4\}$

(3) $\{(x, y): x^2 + y^4 = 17\}$ (4) $\{(x, y): x^4 + y^3 = 17\}$

(d) $(x, y) \rightarrow (x + 2, y + 3)$

(1) $\{(x, y): y = \frac{3}{2}x + 5\}$ (2) $\{(x, y): y = \frac{3}{2}x + 4\}$

(3) $\{(x, y): 3x - 2y + 1 = 0\}$ (4) $\{(x, y): x = 2\}$

12. Suppose U, V, W, and Z are four points in the number plane such that

$$U = (85, 16), \qquad V = (97, -101),$$

$$W = (97, 16), \qquad Z = (85, -101).$$

(a) What is the midpoint of $\overline{UW}$? That is, give the ordered pair which is the midpoint of $\overline{UW}$.

(b) What is the midpoint of $\overline{UZ}$?

(c) Suppose A is the midpoint of $\overline{UW}$, B is the midpoint of $\overline{UZ}$, C is the midpoint of $\overline{ZV}$, and D is the midpoint of $\overline{VW}$. What points if any are in

(1) $\overleftrightarrow{AC} \cap \overleftrightarrow{BD}$? * (2) $\overleftrightarrow{AB} \cap \overleftrightarrow{CD}$?

(3) $\overleftrightarrow{UV} \cap \overleftrightarrow{WZ}$? (4) $\overleftrightarrow{VB} \cap \overleftrightarrow{ZD}$?

☆(d) What points, if any, are in

(1) $\overleftrightarrow{UV} \cap \{(x, y): y = 0\}$? (2) $\overleftrightarrow{ZW} \cap \{(x, y): x = 0\}$?

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*In case you haven't guessed it, "$\overleftrightarrow{AC}$" means the straight line through the points A and C.

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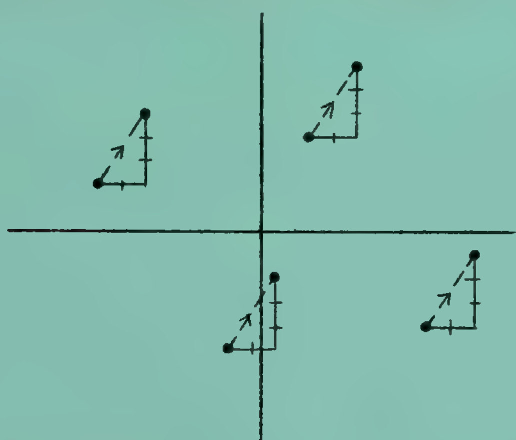
(c) The transformation in (b) belongs to the class of transformations called reflections. The transformation in (c) is a reflection, also. Each dot changes places with its mirror image, the edge of the mirror being placed along the graph of the x-axis. Among the listed sets, those which are closed under this rule are (1) and (3). '$x + 7 = |y|$' is equivalent to '$x + 7 = |-y|$' and '$x^2 + y^4 = 17$' is equivalent to '$x^2 + (-y)^4 = 17$'. [A knowledge of reflections is handy in sketching graphs. For example, in graphing the set selector given in (1), you could concentrate on ordered pairs in quadrants I and II, and just sketch the mirror image below the horizontal axis. In graphing '$x^2 + y^4 = 17$', if you note that $\{(x, y): x^2 + y^4 = 17\}$ is also closed under the rule: $(x, y) \rightarrow (-x, y)$, you could concentrate on plotting points of quadrant I, then reflect in the vertical axis to get the points for quadrant II, and then reflect in the horizontal axis to get the points for quadrants III and IV.]

(d) This transformation [called a translation] may be visualized by thinking of it as moving dots 2 units to the right and 3 units up.



So, straight lines which tilt in this direction are moved along themselves by this transformation. Since the four sets listed have straight line graphs, the students' job is to discover which have slope $\frac{3}{2}$. This will probably be a trial-and-error investigation for most students. Those who have discovered the technique of substituting and seeing if the new sentence is

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equivalent to the old will have lots of fun. For (1), we have:

$$y = \frac{3}{2}x + 5$$
$$y + 3 = \frac{3}{2}(x + 2) + 5$$
$$y + 3 = \frac{3}{2}x + 3 + 5$$
$$y = \frac{3}{2}x + 5 .$$

So, (1) is closed under this rule. Similarly, we know that (2) is closed. By now, students recognize that it helps to solve '$3x - 2y + 1 = 0$' for 'y' to determine if (3) is closed. [But, substitution in the given selector used in naming (3) will yield satisfactory results anyway.]

*

If you are teaching trigonometry to advanced students, you may be able to give them additional insight into the notion of periodicity by giving them questions of the type in Exercise 11. For example:

1. $(x, y) \to (x + 2\pi, y)$

   (a) $\{(x, y): y = \sin x\}$ (b) $\{(x, y): y = \cos x\}$

   (c) $\{(x, y): y = 3 + \sin x\}$ (d) $\{(x, y): y = 3\sin x\}$

2. $(x, y) \to (x + \pi, -y)$

   (a) $\{(x, y): y = \sin x\}$ (b) $\{(x, y): y = \cos x\}$

   (c) $\{(x, y): y = -\sin x\}$ (d) $\{(x, y): y = -\cos x\}$

3. $(x, y) \to (x + 3, y)$

   (a) $\{(x, y): y = \sin x\}$ (b) $\{(x, y): y = \sin \frac{2\pi}{3} x\}$

   (c) $\{(x, y): y = \cos x\}$ (d) $\{(x, y): y = \cos \frac{2\pi}{3} x\}$

*

12. This exercise seeks to make use of the student's geometric intuition and his ability to generalize. The "large" numbers will force him to generalize, and to avoid strictly graphical methods.

    (a) To find the midpoint of $\overleftrightarrow{UW}$, he must first notice that the graph

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of $\overline{UW}$ is a horizontal segment. Then, he finds the first component of the midpoint by computing the average of the first components of the end points. So, the midpoint of $\overline{UW}$ is (91, 16).

(b) The midpoint of $\overline{UZ}$ is (85, −42.5).

(c) Here he visualizes the graph of UWVZ as a rectangle. The graph of $\overline{AC}$ is a vertical segment, and the graph of $\overline{BD}$ is a horizontal segment.

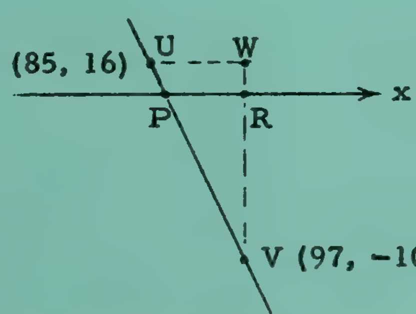
(1) His intuition tells him that $\overline{AC}$ and $\overline{BD}$ bisect each other. This common midpoint is the only element in $\overleftrightarrow{AC} \cap \overleftrightarrow{BD}$. So, (91, −42.5) is the only point in $\overleftrightarrow{AC} \cap \overleftrightarrow{BD}$.

(2) By intuition, the student knows that $\overleftrightarrow{AB} \cap \overleftrightarrow{CD} = \emptyset$. So, his answer is 'none'.

(3) Again by intuition, he knows that the diagonals of a rectangle bisect each other and that they also bisect the segments joining the midpoints of opposite sides. So, the only point in $\overleftrightarrow{UV} \cap \overleftrightarrow{WZ}$ is (91, −42.5).

(4) The student recognizes that the graph of VZBD is a rectangle. Generalizing from (1) and (3), he finds that the only point in $\overleftrightarrow{VB} \cap \overleftrightarrow{ZD}$ is (91, −71.75).

☆(d) This exercise is a "stretcher", but able students will be able to solve it if you give them an opportunity. The student makes a rough sketch like this for (1).



Then he reasons as follows. To go from V to U by way of W, I would go up 117 units and over 12. To go from V to P by

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of R, I would go up 101 units and over not as far as from W to U. In fact, to go from V to R, I went $\frac{101}{117}$ of the distance between V and W. So, to go from R to P, I should go $\frac{101}{117}$ of the distance between W and U.

$$\frac{101}{117} \times 12 = 10\frac{14}{39}$$

So, the x-coordinate of the graph of P is $10\frac{14}{39}$ less than the x-coordinate of the graph of V. Hence,

$$\overleftrightarrow{UV} \cap \{(x, y): y = 0\} = \{(86\tfrac{25}{39}, 0)\}.$$

A similar slope-type argument would lead to the fact that

$$\overleftrightarrow{ZW} \cap \{(x, y): x = 0\} = \{(0, -929\tfrac{3}{4})\}.$$

[A student might complain that he could do this problem if the numbers weren't so hard. Suggest to him that he give himself easier numbers in order to work out a "theory" and then apply the theory to the given problem.]

This problem might provoke a student into trying to find an equation whose graph is the line determined by two given points. A helpful start for such a student might be the consideration of the fact that the graph of each equation of the form:

$$A(x - 85) + B(y - 16) = 0,$$

where not both A and B are 0, is a straight line which passes through the graph of U. His job, then, is to find values of 'A' and 'B' such that when 'A' and 'B' are replaced by names of these values, the resulting equation is satisfied by the components of V. This should be enough of a hint for any student who is bright enough to tackle (d).

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(2) 3

(3) 1, 3, 4, 5, 6, 7, 9, 11

(4) 3

(5) 1, 3, 5, 7, 9, 11

(6) 1, 3, 5, 7

(7) 1, 3, 5, 7, 9, 11

(8) none [$B \cap \emptyset = \emptyset$]

(9) −3, −2, −1, 0, 1, 2, 3, 4, 5, 6, 7, 9, 11

2. [The note of caution about giving answers for Exercise 1 applies here.]

   (1) none [There are no ordered pairs which belong to both set A and set B.]

   (2) (−2, 3)

   (3) (0, 0)

   (4) (−1, 4)

   (5) (0, 0), (1, −9), (2, −13), (0, −5), (−1, −1), (−2, 3)

   (6) (−1, −1), (0, 0)

   (7) (0, 0), (−1, +4), (−2, 8), (3, −12)

   (8) (−1, −1)

   (9) (−2, 3), (0, 0), (−1, 4), (−2, 8), (3, −12)

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B. 1. (a) $\frac{1}{12}$ (b) $\frac{1}{12}$ (c) $\frac{1}{6}$ (d) $\frac{1}{4}$

(e) 0 (f) $\frac{1}{6}$ (g) $\frac{1}{2}$

2. (a) $\frac{1}{6}$ (b) $\frac{1}{12}$ (c) $\frac{1}{18}$ (d) $\frac{1}{18}$ (e) $\frac{1}{36}$

(f) 0 (g) $\frac{1}{12}$ (h) 0 (i) $\frac{5}{36}$ (j) $\frac{1}{4}$

C. [The purpose of Part C is to provide practice in reading graphs.]

1. (1) (0, 0) (2) (3, 8) (3) (1, 3) (4) (−1, 4)

(5) $(4\frac{1}{2}, 3)$ (6) $(4\frac{1}{2}, 3)$ (7) (−1, 4) (8) (−1, 4)

(9) (−1, 4) and (1, 3) (10) (1, 3) and (−1, 4)

2. (1) c (2) b (3) a (4) d (5) g (6) none.

3. (1) (3, 8) (2) (3, 9) (3) (−1, 4)

(4) (1, 3) (5) (−1, 4)

D. 1. (1) 1, 3 [If asked to describe the set $A \cap B$ by listing its elements, one would write '{1, 3}', [or, perhaps, '$A \cap B = \{1, 3\}$'], or say 'the set consisting of 1 and 3'. To carry out the directions of Part D ['Tell the elements of the sets listed below'], one lists just the numbers 1 and 3, or says '1 and 3'. It would be incorrect to write:

$$A \cap B = 1, 3$$

since the intersection of set A and set B is itself a set, and it would be equally incorrect to say 'A intersection B is 1 and 3'. It would be correct to say 'the elements of A intersection B are 1 and 3'.]

B. 1. Suppose you have a die and also a "coin" which has a numeral '0' on one side and a numeral '1' on the other. By throwing the die you can get a number from $\{1, 2, 3, 4, 5, 6\}$, and by flipping the coin you can get a number from $\{0, 1\}$. The result of a single throw of both the die and the coin can be interpreted as an ordered pair in the cartesian product

$$\{1, 2, 3, 4, 5, 6\} \times \{0, 1\}.$$

Give the probability of getting, in one throw,

(a) $(2, 0)$. (b) $(2, 1)$.

(c) an element [member] of $\{(5, 0), (6, 1)\}$.

(d) an element of $\{(1, 1), (3, 0), (2, 1)\}$.

(e) $(1, 2)$.

(f) an ordered pair whose first component is 4.

(g) an ordered pair whose second component is 0.

2. Consider the equation:

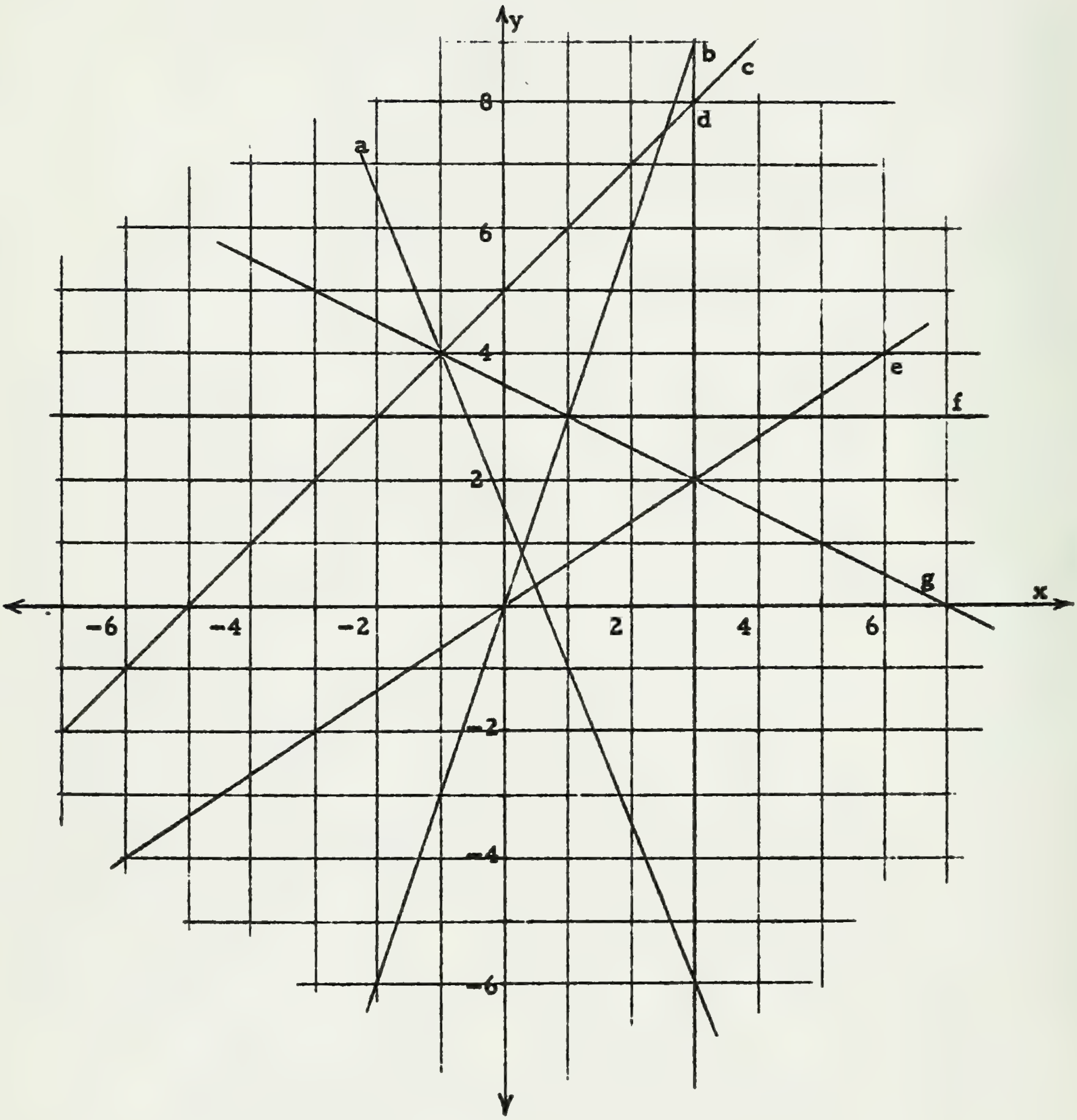
$$(*) \quad Rx = G.$$

Suppose you throw a pair of dice [one red and one green] to get an ordered pair of numbers, agreeing that the red die gives the first component of the ordered pair. If you replace the 'R' in (*) by a name for the first component of the ordered pair, and the 'G' by a name for the second component, you get an equation whose root can be found. What is the probability that the root will be

(a) 1 (b) 2 (c) 3 (d) $\frac{1}{3}$

(e) 6 (f) 7 (g) $\frac{1}{2}$ (h) 0

(i) an even number (j) an odd number


y
x
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-2
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-6
-6
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2
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6

C. 1. Seven sets (straight lines) are pictured on the opposite page. List the points which belong to the sets listed below.

(1) $e \cap b$ [Ans: (0, 0)]	(2) $d \cap c$
(3) $g \cap (b \cap f)$	(4) $c \cap a$
(5) $f \cap e$	(6) $e \cap f$
(7) $g \cap a$	(8) $c \cap g$
(9) $g \cap (a \cup b)$	(10) $(b \cap g) \cup (a \cap g)$

2. In the blank at the left of each of the following equations, write the letter which is the name of its locus.

______(1) $y = 5 + x$	______(2) $3x - y = 0$
______(3) $5x = 3 - 2y$	______(4) $x = 3$
______(5) $y = -\frac{1}{2}x + 3\frac{1}{2}$	______(6) $4y - 6 - 10x = 0$

3. Use the picture to tell which ordered pairs satisfy both equations.

(1) $\left.\begin{array}{l} y = 5 + x \\ x = 3 \end{array}\right\}$

(2) $\left.\begin{array}{r} 3x - y = 0 \\ x = 3 \end{array}\right\}$

(3) $\left.\begin{array}{r} 5x = 3 - 2y \\ y = -\frac{1}{2}x + 3\frac{1}{2} \end{array}\right\}$

(4) $\left.\begin{array}{l} y = 3x \\ y = -\frac{1}{2}x + 3\frac{1}{2} \end{array}\right\}$

(5) $\left.\begin{array}{r} y - x = 5 \\ y + \frac{1}{2}x = 3\frac{1}{2} \end{array}\right\}$

D. 1. Suppose A, B, and C are sets of numbers such that

$A = \{1, 3, 5, 7, 9, 11\}$,

$B = \{-3, -2, -1, 0, 1, 2, 3\}$, and

$C = \{3, 4, 5, 6, 7\}$.

Tell the elements of the sets listed below.

(1) $A \cap B$	(2) $B \cap C$	(3) $C \cup A$
(4) $C \cap B$	(5) $A \cup (B \cap C)$	(6) $A \cap (B \cup C)$
(7) $A \cup \emptyset$	(8) $B \cap \emptyset$	(9) $(A \cup B) \cup C$

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2. Given these sets of ordered pairs

A: $\{(1, -9), (2, -13), (0, -5), (-1, -1), (-2, 3)\}$,

B: $\{(0, 0), (-1, 4), (-2, 8), (3, -12)\}$,

C: $\{(-1, -1), (0, 0), (2, 2), (4, 4), (6, 6)\}$,

D: $\{(0, 5), (1, 6), (3, 8), (-2, 3), (-1, 4), (-5, 0)\}$,

determine the ordered pairs in the sets listed below.

(1) $A \cap B$ (2) $A \cap D$ (3) $B \cap C$

(4) $B \cap D$ (5) $A \cup (B \cap C)$ (6) $(A \cup B) \cap C$

(7) $(C \cap D) \cup B$ (8) $A \cap (B \cup C)$ (9) $B \cup (A \cap D)$

E. Make labeled drawings of these loci.

1. the locus in (m, n) of '$2m - n = 8$'
2. the locus in (a, b) of '$3a + 2b = 10$'
3. the locus in (c, d) of '$c \geq 2d$'
4. the locus in (p, q) of '$2q - 10 = 3p$'
5. the locus in (r, s) of '$8 + |r| = 2s$'.

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[Recall the convention explained on page 4-31 regarding sentences which contain the pronumerals 'x' or 'y'; then graph the sentences which follow.]

6. $-2y = x + 5$
7. $y = -x + 3$
8. $x \geq y + 2$
9. $y \leq -3x - 1$
10. $2x + 3y - 2 = 0$
11. $y + 5 > -2x$
12. $|x| + |y| \geq 6$
13. $|x| + |y| \leq 6$ or $|x| - |y| \geq 6$

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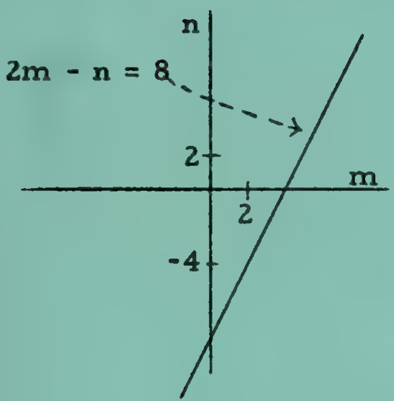
E. 1.



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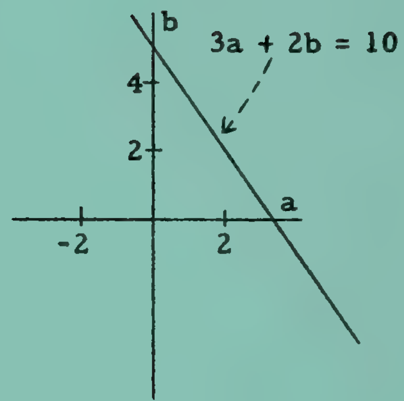

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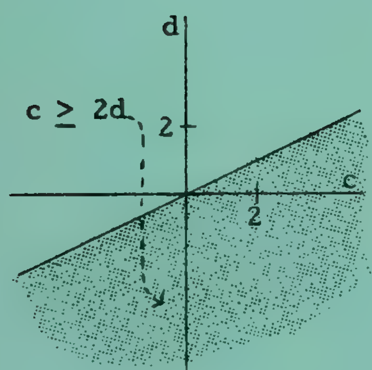

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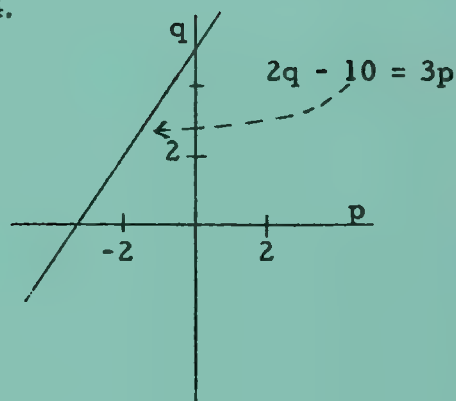

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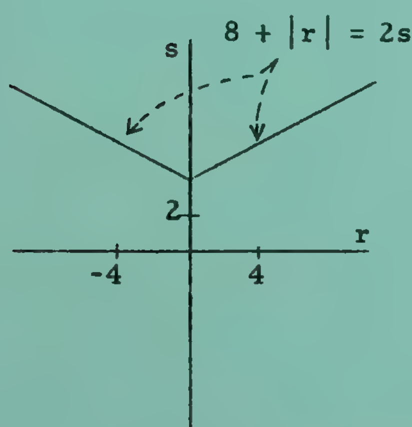

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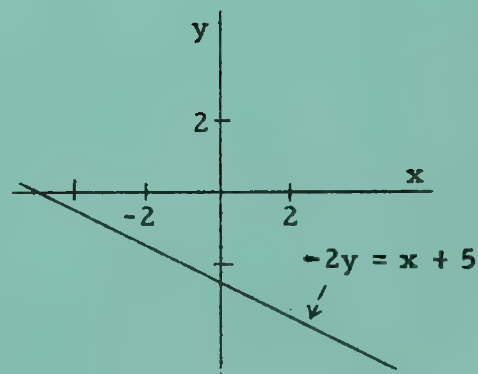
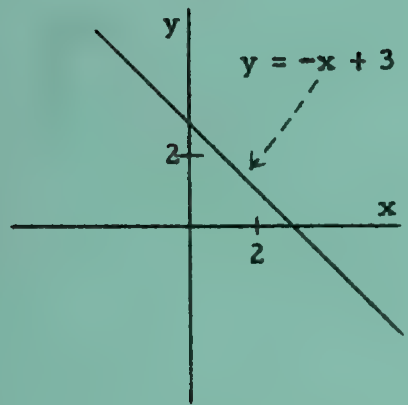


[4-94]

low.

13}

7.



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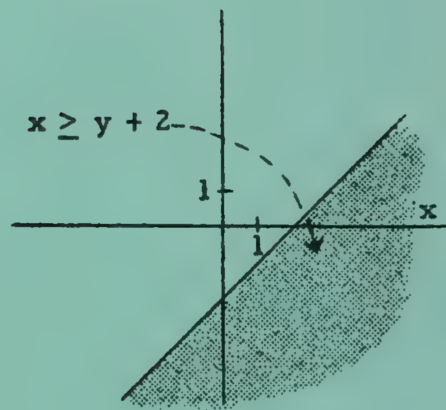

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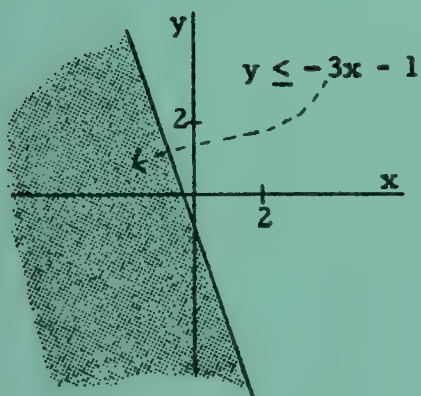

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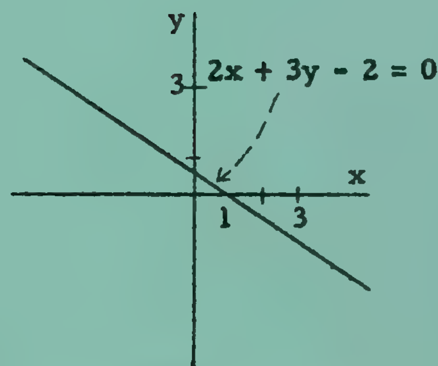

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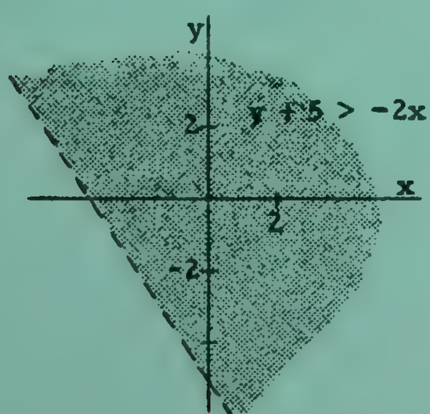

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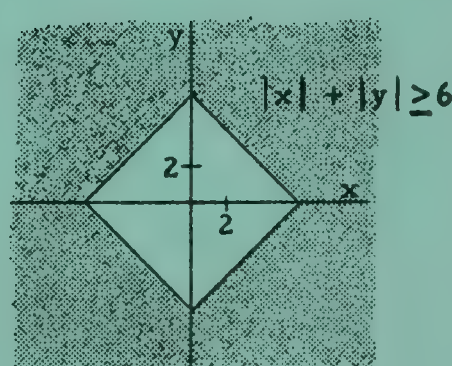


[Exercise 13 is answered on TC[4-95]a.]

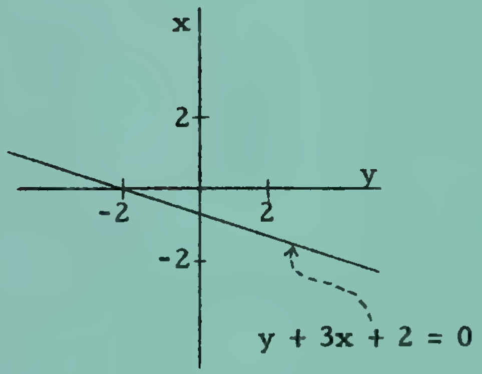
[4-94]

[4-95]

low.

· 13}

19.



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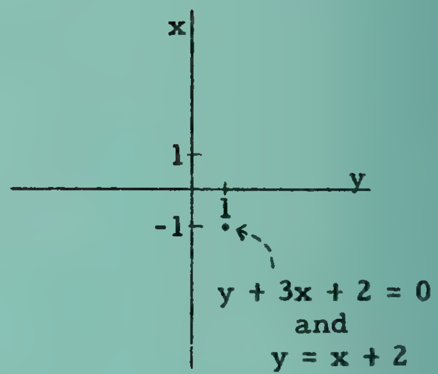

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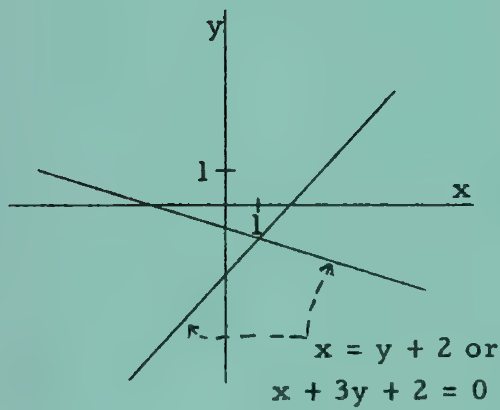

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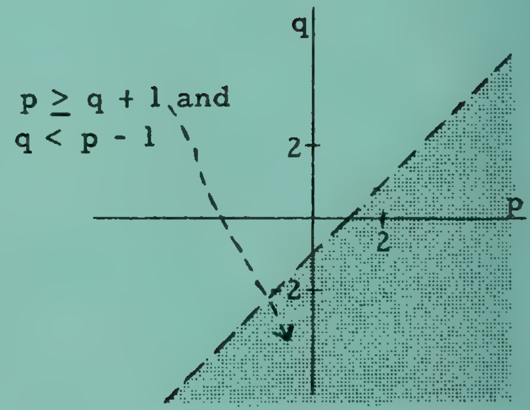


F. 1. $-14a - 13$ 2. $-14cx$ 3. $50 - 71m$ 4. 347

5. $20a + ab$ 6. $9d + x$ 7. $5a - 117b + 9$

8. $-2a^2 - 9ab - 3b^2$ 9. $m^2 - 9mn + 9n^2 + 2n + 30$

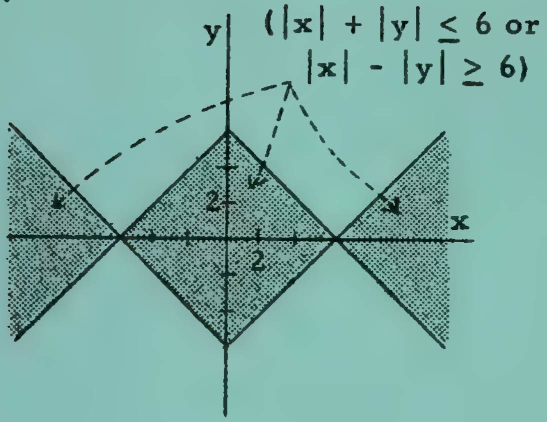
10. $14t^2 + 33t - 30$ 11. $3x^2 - 15x + 6y^2 + p^2$

[4-95]

low.

13}

13.



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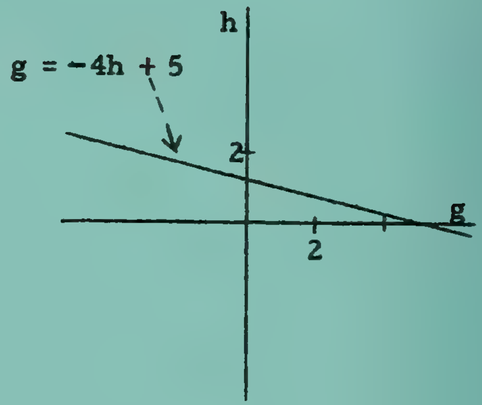

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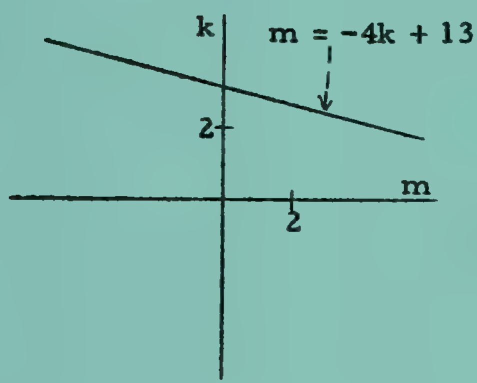

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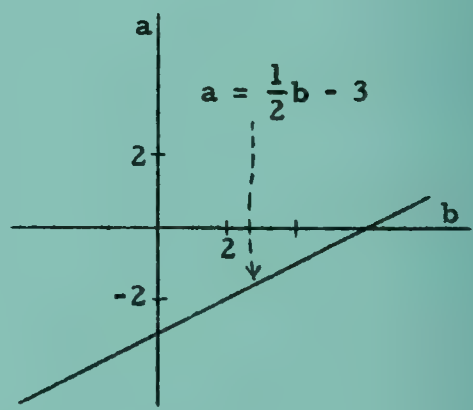

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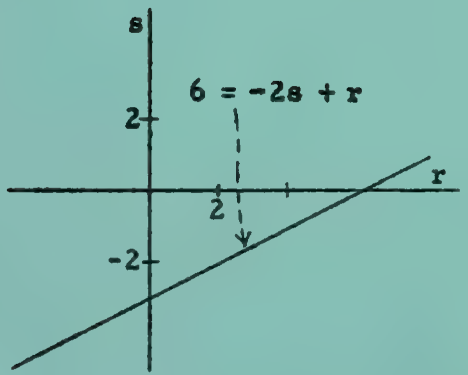

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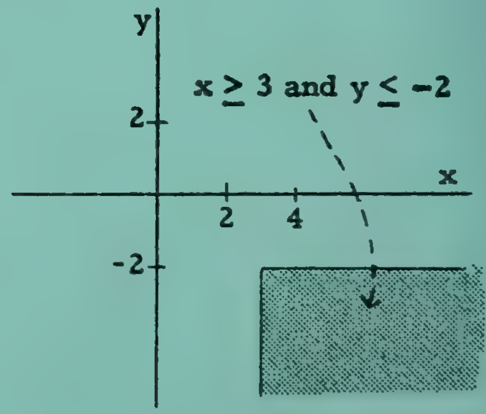


On a picture of the number plane, graph the sets listed below.

14. $\{(g, h): g = -4h + 5\}$

15. $\{(m, k): m = -4k + 13\}$

16. $\{(b, a): a = \frac{1}{2}b - 3\}$

17. $\{(r, s): 6 = -2s + r\}$

18. $\{(x, y): x \geq 3 \text{ and } y \leq -2\}$

19. $\{(y, x): y + 3x + 2 = 0\}$

20. $\{(y, x): y + 3x + 2 = 0 \text{ and } y = x + 2\}$

21. $\{(x, y): x = y + 2 \text{ or } x + 3y + 2 = 0\}$

22. $\{(p, q): p \geq q + 1 \text{ and } q < p - 1\}$

F. Fill in the blanks with the simplest expressions to make true sentences.

1. For each a, the sum of $3a + 2$ and $-17a - 15$ is ________.

2. For each c, for each x, the sum of cx and $-15cx$ is ________.

3. For each m, the sum of $32 + m$ and $18 - 72m$ is ________.

4. For each r, the sum of $\sqrt{16} + r\sqrt{25}$ and $-5r + 343$ is ________.

5. For each a, for each b, the sum of $3a - ab + b$ and $2ab - b + 17a$ is ________.

6. For each h, for each d, for each x, the sum of $hx + 2d - x + dx$ and $(2 - h)x + 7d - dx$ is ________.

7. For each a, for each b, the sum of $3a - 2b + 7$, $-15b + 2 + a$, and $a - 100b$ is ________.

8. For each a, for each b, if I subtract $a^2 + 2ab + b^2$ from $-a^2 - 7ab - 2b^2$, I get ________.

9. For each m, for each n, the sum of $2m^2 + mn - n^2 + m - n + 10$ and $-m^2 - 10mn + 10n^2 - m + 3n + 20$ is ________.

10. For each t, the sum of $3t^2 + 2t - 10$, $-4t^2 - t - 15$, and $15t^2 + 32t - 5$ is ________.

11. For each x, for each y, for each p, the sum of $3x^2 + 2y^2 - p^2$ and $4y^2 + 2p^2 - 15x$ is ________.

(continued on next page)

12. For each x, for each y, the sum of $43x^2 + 2xy - y^2 + 1$ and $x - xy + y + 2$ is ___________.

13. For each a, for each c, the difference of $a^2 + ac - 2c^2$ from the sum of $3a^2 - 5ac + 2c^2$ and $a^2 + 7ac - 3c^2$ is ___________.

14. For each a, for each b, the difference of $14a + 15b$ from $10a + 6b$ is _________.

15. For each x, for each m, subtracting $x^2 + mx$ from $x^2 - mx + 2$ gives ___________.

16. For each p, if I subtract $-17p$ from -1, I get ___________.

17. For each r, the difference of $3r^2 - 10r + 15$ from $17r^2 + 17r - 5$ is ___________.

18. For each h, for each j, for each k, the difference of $h - j + k$ from the sum of $2h - j$ and $j + k$ is ___________.

19. For each x, for each y, the difference of $x^2 + xy - y^2 - 2x - y + 15$ from $2x^2 - 30xy + y^2 + x - 10$ is ___________.

20. For each r, the product of $2r + 1$ by $-2r^3$ is ___________.

21. For each a, for each b, the product of $a^2 + 2ab + b^2$ by $-3ab$ is ___________.

22. For each g, for each k, the product of $14gk$ by $-14g + k + gk - 53$ is ___________.

23. For each p, for each y, for each t, the product of $-py + yt + t - p$ by pyt is ___________.

24. For each b, the product of $7b - 1$ by $7b + 4$ is ___________.

25. For each c, the product of $-c + 2c - 3$ by $-4c$ is ___________.

26. For each $a \neq 0$, for each $b \neq 0$, the quotient of ab by ab is ________.

27. For each $a \neq 0$, for each $c \neq 0$, the quotient of $3a^2c$ by ac^2 is ___________.

28. For each $r \neq 0$, for each $s \neq 0$, for each $t \neq 0$, the quotient of $32\ rst^2$ by $2rst$ is __________.

4/150

12. $43x^2 + xy - y^2 + x + y + 3$ 13. $3a^2 + ac + c^2$

14. $-4a - 9b$ 15. $-2mx + 2$ 16. $17p - 1$

17. $14r^2 + 27r - 20$ 18. $h + j$

19. $x^2 - 31xy + 2y^2 + 3x + y - 25$ 20. $-4r^4 - 2r^3$

21. $-3a^3b - 6a^2b^2 - 3ab^3$ 22. $-196g^2k + 14gk^2 + 14g^2k^2 - 742gk$

23. $-p^2y^2t + py^2t^2 + pyt^2 - p^2yt$ 24. $49b^2 + 21b - 4$

25. $-4c^2 + 12c$ 26. 1 27. $(3a)/c$

28. $16t$ 29. $6cm^2$ 30. $-15/2$

31. $a^2 + a - 2$ 32. $45a^2 - 27ab - 2b^2$

33. $(x^2 - 2xy + y^2)/2$ 34. $12m^2 - 180m + 675$

35. $2x^2 + 8xt + 8t^2$ 36. $3S^2 - 2S^2V - V^2S^2$

37. $3a^2b^2 - 8abc - 3c^2$ 38. $12r^2t^2 + 37rtaw - 30a^2w^2$

39. $(a + b)^2 + 2c(a + b) + c^2$ [or: $a^2 + 2ab + b^2 + 2ac + 2bc + c^2$]

40. $16p^2 - 2p$ 41. ab

42. $\frac{a + 2b + 4b^2}{2}$ [or: $\frac{a}{2} + b + 2b^2$] 43. $4j^2 - \frac{4}{3}jk + \frac{1}{9}k^2$

44. $x_0^2 + 2x_0x_1 - 15x_1^2$ 45. $-75yz$

46. -1 47. $-m^2 + mn + 2n^2 - m + 2n$

48. $\frac{6}{b^2} - \frac{5}{bc} - \frac{4}{c^2}$ 49. $4m^2 + \frac{m}{d} - \frac{3}{d^2}$

50. $-3r^2 - 4rt - t^2 - 4r - 2t - 1$ 51. $1.43 + x$ 52. x

G. 1. 14 2. 13 3. $\sqrt{5}$ 4. $\frac{3}{4}$

5. $\frac{63}{250}$, [or: .252] 6. $\frac{1}{\pi} + 1$ [or: $\frac{1 + \pi}{\pi}$] 7. $\frac{1792}{45}$

12. Fo

13

29. For each $k \neq 0$, for each $m \neq 0$, for each c, the quotient of $2km^2 \cdot 3cm$ by km is __________.

30. For each $n \neq 0$, for each $m \neq 0$, the quotient of $15n^2 \cdot m \cdot m$ by $-2m^2n^2$ is __________.

31. For each a, the product of $a + 2$ by $a - 1$ is __________.

32. For each a, for each b, the product of $3a - 2b$ by $15a + b$ is __________.

33. For each x, for each y, the product of $x - y$ by one half of itself is __________.

34. For each m, the product of $2m - 15$ by three times itself is __________.

35. For each x, for each t, the product of $2x + 4t$ by $x + 2t$ is __________.

36. For each S, for each V, the product of $3S + VS$ by $S - VS$ is __________.

37. For each a, for each b, for each c, the product of $3ab + c$ by $ab - 3c$ is __________.

38. For each r, for each w, for each t, for each a, the product of $3rt - 2aw$ by $4rt + 15aw$ is __________.

39. For each a, for each b, for each c, the product of $a + b + c$ by itself is __________.

40. For each p, the sum of $-\frac{1}{16}$ and the square of $4p - \frac{1}{4}$ is ________.

41. For each a, for each b, the sum of $a^2 - ab + b^2$ and the product of $-a + b$ by $a - b$ is __________.

42. For each $a \neq 0$, for each b, the quotient of $4a^2 + 8ab + 16ab^2$ by 8a is __________.

43. For each j, for each k, the product of $2j - \frac{1}{3}k$ by itself is ________.

44. For each x_0, for each x_1, the product of $x_0 - 3x_1$ by $x_0 + 5x_1$ is __________.

(continued on next page)

45. For each y, for each z, the difference of $3y^2 + yz - 25z^2$ from the product of $3y + z$ by $y - 25z$ is ____________.

46. For each a, for each $z \neq a$, the quotient of $a - z$ by $z - a$ is ____________.

47. For each m, for each n, the product of $m + n + 1$ by $-m + 2n$ is ____________.

48. For each $b \neq 0$, for each $c \neq 0$, the product of $\frac{3}{b} - \frac{4}{c}$ by $\frac{2}{b} + \frac{1}{c}$ is ____________.

49. For each $d \neq 0$, for each m, the product of $\frac{1}{d} + m$ by $4m - \frac{3}{d}$ is ____________.

50. For each r, for each t, the product of $3r + t + 1$ by $-r - t - 1$ is ____________.

51. For each x, ____________ exceeds x by 1.43.

52. For each y, for each x, y increased by the difference of y from x is ____________.

G. Evaluate each of the following pronumeral expressions using the given values of the pronumerals.

1. $\sqrt{2as}$; '14' for 'a', '7' for 's'.

2. $\sqrt{9a^2 - 6ab + b^2}$; '−1' for 'a', '10' for 'b'.

3. $\sqrt{\dfrac{3 + a}{a - 1}}$; '2' for 'a'.

4. $\sqrt{\dfrac{a - 2\sqrt{b} + 1}{a + b}}$; '4' for 'b', '12' for 'a'.

5. $\frac{h}{2}(a^{-3} + A^{-3})$; '4' for 'h', '2' for 'a', '10' for 'A'.

6. $\pi r^2 + ar$; '$\frac{1}{\pi}$' for 'r', 'π' for 'a'.

7. $\dfrac{\frac{1}{2}ax + \frac{1}{3}bx}{1 + 2}$; '2.2' for 'a', '7.9' for 'b', '32' for 'x'.

[4-99]

'9' for 't'.

8. 130 9. 9/4 10. 2.01π

H. 1. $v = 2t - sw$, $[w \neq 0]$ 2. $w = \frac{2t - v}{s}$, $[w \neq 0 \neq s]$ 3. $t = \frac{1}{9}r$

4. $p = \frac{3mn}{2n + m}$, $[n \neq 0 \neq m \neq -2n, \;\; p \neq 0]$

5. $B = \frac{2A - ah}{h}$, $[h \neq 0]$ 6. $a = \frac{2s - 2s_0 - 2vt}{t^2}$, $[t \neq 0]$

7. $y = \frac{9}{a - 9}$, $[0 \neq y \neq -\frac{1}{2}, \; a \neq 9]$

8. $y = \frac{a - 9}{9 - 3a}$, $[0 \neq y \neq -\frac{1}{2}, \; a \neq 3]$

9. $b = \frac{-c - ax^2}{x}$, $[x \neq 0]$ 10. $f = \frac{15 - de - ac - ab}{d}$, $[d \neq 0]$

I. 1. 1, 2, 4, 7, 14, 28

2. 1, −1, 2, −2, 4, −4, 7, −7, 14, −14, 28, −28

3. 1, −1, 2, −2, 4, −4, 7, −7, 14, −14, 28, −28

4. None 5. 0, 0 6. 1, −1

7. 4[1, 2, 4], 9[1, 3, 9], 25[1, 5, 25], 49[1, 7, 49]

8. None [For each positive integer which is a factor of any number with respect to the set of integers, there is a negative integer which is also a factor of that number with respect to the set of integers.]

9. 101, 103, 107, 109

10. $2^7 \cdot 3^3 \cdot 5 \cdot 11$

8. $s_0 + v_0t + \frac{1}{2}at$; '4' for 's_0', '-2' for 'v_0', '32' for 'a', '9' for 't'.

9. $-\frac{1}{p} + \frac{1}{2} \cdot \frac{pq^2}{p + q^2}$; '$-\frac{1}{2}$' for 'p', '$\frac{1}{2}$' for 'q'.

10. $\pi\ell(r_1 + r_2)$; '1' for 'ℓ', '2' for 'r_1', '.01' for 'r_2'.

H. Solve these equations for the pronumeral indicated.

1. $s = \frac{2t - v}{w}$; v

2. $s = \frac{2t - v}{w}$; w

3. $7t - \frac{1}{3}r = r - 5t$; t

4. $\frac{2}{m} + \frac{1}{n} = \frac{3}{p}$; p

5. $A = \frac{1}{2}h(a + B)$; B

6. $s = s_0 + vt + \frac{1}{2}at^2$; a

7. $\frac{9}{y} = \frac{9 + a}{2y + 1}$; y

8. $\frac{9 - a}{y} = \frac{9 + a}{2y + 1}$; y

9. $ax^2 + bx + c = 0$; b

10. $a(b + c) + d(e + f) - 15 = 0$; f

I. 1. What are all the factors of 28 with respect to the set of positive integers?

2. What are all the factors of 28 with respect to the set of integers?

3. What are all the factors of -28 with respect to the set of integers?

4. What are all the factors of -28 with respect to the set of negative integers?

5. What numbers are not factors of 28 with respect to the set of rationals? The set of reals?

6. What are the factors of -1 with respect to the set of integers?

7. Give 4 numbers which have exactly 3 factors with respect to the set of positive integers.

8. Give 1 number which has exactly 5 factors with respect to the set of integers.

9. List all the prime numbers which are greater than 100 but less than 110.

10. Give the prime power factorization of 190080.

J. Factor.

1. $4x - 4y$	2. $a^2x - bx^2$
3. $x^2 - 3x - 18$	4. $d^2 + 7d + 10$
5. $x^2 - 10x + 16$	6. $t^2 - 49$
7. $x^2 - 16x + 64$	8. $5m^2 + 3m - 2$
9. $6x^2 + 7x - 3$	10. $8x^2 + 26x - 45$
11. $4r^2 - 20r + 25$	12. $16 - 24p + 9p^2$
13. $16 + 29x - 6x^2$	14. $5z^2 - 45$
15. $8y^2 + 12y - 36$	16. $24x^2 + 24x + 6$
17. $x^4 + 5x^2 + 4$	18. $k^4 - 1$
19. $x^4 - 7x^2 - 18$	20. $2x^3 - 11x^2 + 12x$
21. $(a - 3)^2 - (a + 4)^2$	22. $4x^2 - 9y^2$
23. $x^2 - 7xy + 12y^2$	24. $6x^2 - 11xy - 10y^2$
25. $x^4y - 7x^3y + 12x^2y$	26. $8x^3 - 40x^2y + 50xy^2$
27. $-50x + 28 + 12x^2$	28. $a^2c + a^3 - 12ac^2$
29. $(x^2 - 3)^2 - (x^2 - 5)^2$	30. $c^5d^5 + 36cd - 13c^3d^3$

K. For each of the following problems, make a graph which could be used to solve problems of that type, and solve the problem.

1. A taxicab company charges its customers at a rate of 35 cents for the first $\frac{2}{3}$ mile traveled, and 10 cents for each additional $\frac{1}{3}$ mile or fraction thereof. How much is the fare for a trip of 5 miles?

2. Parcels which are sent at "book rate" cost 9 cents for the first pound and 5 cents for each additional pound or fraction thereof. How much would it cost to ship a box of books which weighs 12 pounds?

J. 1. $4(x - y)$ 2. $x(a^2 - bx)$ 3. $(x - 6)(x + 3)$

4. $(d + 5)(d + 2)$ 5. $(x - 8)(x - 2)$ 6. $(t - 7)(t + 7)$

7. $(x - 8)(x - 8)$ 8. $(5m - 2)(m + 1)$ 9. $(3x - 1)(2x + 3)$

10. $(4x - 5)(2x + 9)$ 11. $(2r - 5)^2$ [or: $(2r - 5)(2r - 5)$]

12. $(4 - 3p)^2$ 13. $(16 - 3x)(1 + 2x)$ 14. $5(z - 3)(z + 3)$

15. $4(2y - 3)(y + 3)$ 16. $6(2x + 1)^2$

17. $(x^2 + 4)(x^2 + 1)$ 18. $(k^2 + 1)(k + 1)(k - 1)$

19. $(x - 3)(x + 3)(x^2 + 2)$ 20. $x(2x - 3)(x - 4)$

21. $-7(2a + 1)$ 22. $(2x - 3y)(2x + 3y)$

23. $(x - 3y)(x - 4y)$ 24. $(3x + 2y)(2x - 5y)$

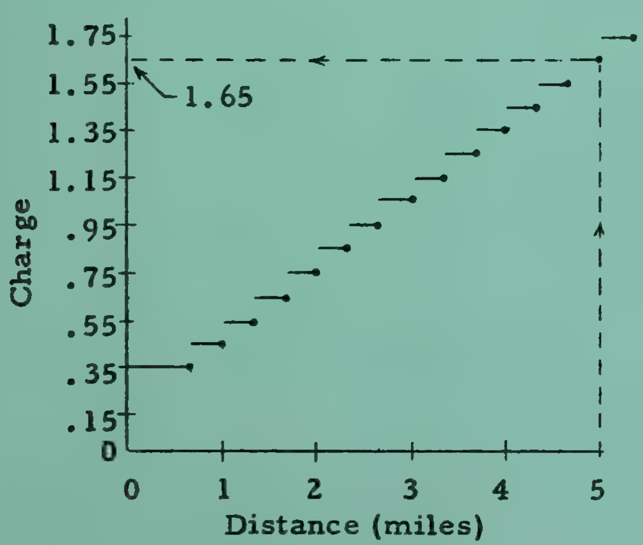
25. $x^2y(x - 3)(x - 4)$ 26. $2x(2x - 5y)^2$

27. $2(7 - 2x)(2 - 3x)$ [or: $2(2x - 7)(3x - 2)$]

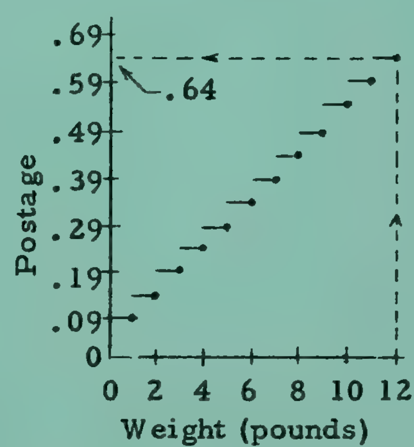
28. $a(a + 4c)(a - 3c)$ 29. $4(x + 2)(x - 2)$

30. $cd(cd - 3)(cd + 3)(cd - 2)(cd + 2)$ [or: $cd(3 - cd)(3 + cd)(2 - cd)(2 + cd)$]

K. 1.



2.



[4-100]

[4-101]

ɔf

L.

1.	5	2.	-2	3.	4	4.	12
5.	$\{x: x > -2\}$	6.	1	7.	6	8.	$\{x: x < 2\}$
9.	$\frac{1}{3}$	10.	no roots	11.	$-8, 11$	12.	$-\frac{15}{7}, \frac{11}{7}$
13.	-5	14.	2	15.	2.1	16.	2
17.	1	18.	each real number is a root			19.	no roots
20.	0	21.	-13	22.	4	23.	$\{x: x \leq 1\}$
24.	$-\frac{3}{16}$	25.	$12, -1$	26.	$\frac{1}{3}$	27.	$-\frac{9}{2}$
28.	$2, -11$	29.	$-\frac{1}{7}, -1$	30.	$4, -4$	31.	1
32.	$\frac{12}{5}$	33.	$\frac{7}{12}$	34.	-31	35.	2
36.	$3, -2$	37.	$2, -\frac{3}{4}$	38.	no roots	39.	0, 5
40.	$2, -12$	41.	$-\frac{37}{2}$	42.	-1	43.	no roots
44.	6	45.	$\frac{3}{2}$	46.	1	47.	-1
48.	-1						

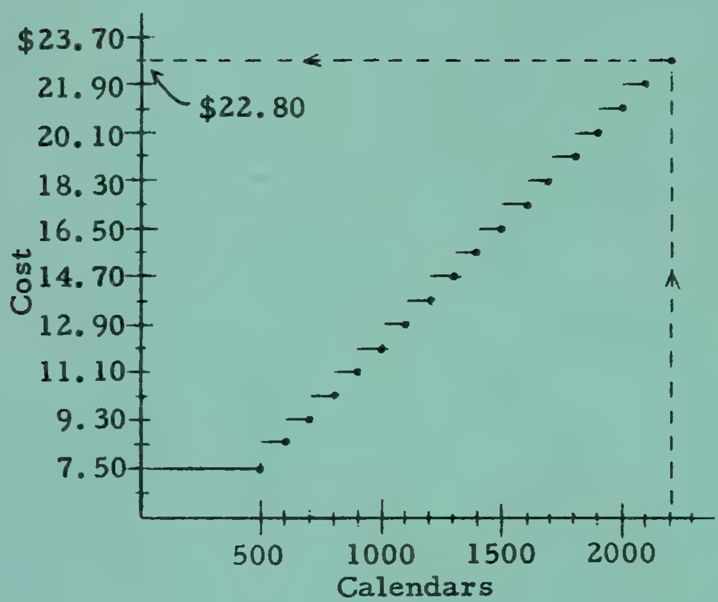
M.

1.	$7x + 4$	2.	$-4z - 1$	3.	$5a - 8$
4.	$9a - 10$	5.	$8d - 6$	6.	$6x - 4$
7.	$4y - 1$	8.	$7a - 10b$	9.	$-x^4 + x^2 - x$
10.	$a + b + 6$	11.	$-x^3 + 7x^2 - 6x$	12.	$x^2 + 3x - 14$
13.	$-2x^3y + 3x^4y^3$	14.	$6a^4b^3 - 12a^2b^4$	15.	$-3x^2 + 7x - 10$
16.	0	17.	$-12a$	18.	$3ab$ $[ab \neq 0]$
19.	$3a^2 - 2$, $[ab \neq 0]$	20.	$\frac{3}{2}xy$, $[xy \neq 0]$	21.	2, $[xy \neq 0]$
22.	$-\frac{x}{y^6}$, $[xyz \neq 0]$	23.	$4 + 3x$, $[x \neq 0]$	24.	$9xy^2 - 2y$, $[x \neq 0]$
25.	$\frac{17}{12x}$, $[x \neq 0]$	26.	$\frac{5x - 17}{x^2 - 7x + 12}$, $[3 \neq x \neq 4]$		

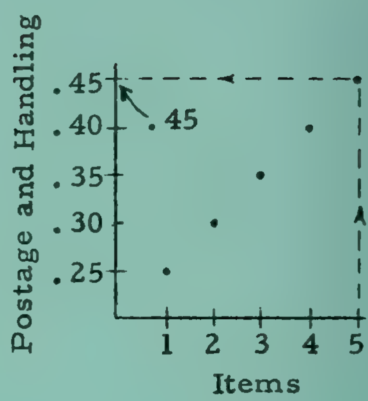
[4-101]

ɔf

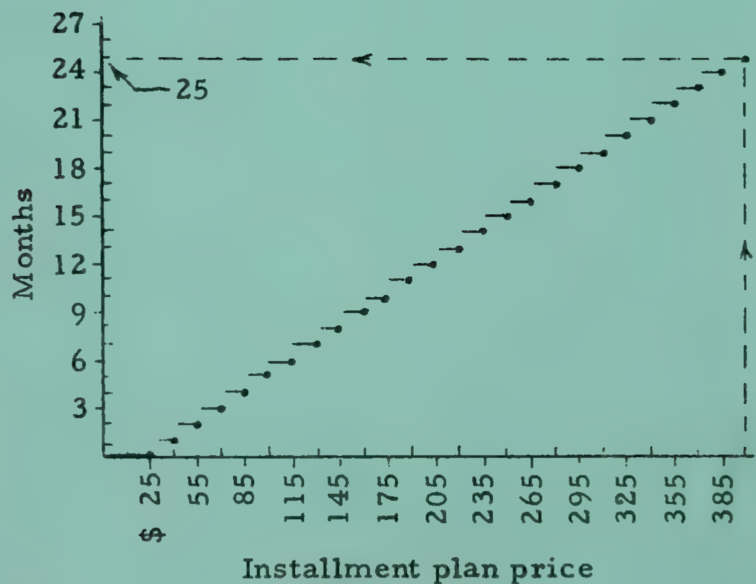
3.


$23.70
21.90
20.10
18.30
16.50
14.70
12.90
11.10
9.30
7.50
$22.80
Cost
500
1000
1500
2000
Calendars

4.


.45
.40
.35
.30
.25
.45
Postage and Handling
1
2
3
4
5
Items

5.


27
24
21
18
15
12
9
6
3
25
Months
$ 25
55
85
115
145
175
205
235
265
295
325
355
385
Installment plan price

3. A printing company will print pocket calendars at a rate of \$7.50 for the first 500 calendars, and 90 cents for each additional 100. How much would 2200 calendars cost at this rate?

4. A department store which accepts mail orders for gift items charges 25 cents for postage and handling on "one item" orders, and on orders for two or more items the charge is 25 cents for the first item and 5 cents for each additional item. How much would be charged for postage and handling on a gift order for 5 items?

5. A store which specializes in selling merchandise on the installment plan advertises refrigerators for "\$25 down and \$15 per month". How many months will be required to complete the payment on a refrigerator for which the "installment plan" price is \$399.95?

L. Solve.

1. $6 + z = 11$
2. $5 - x = 7$
3. $2x - 5 = 3$
4. $3 - \frac{1}{3}x = -1$
5. $5t + 6 > 2t$
6. $6(3 - x) = 12x$
7. $\frac{x}{2} - 2 = \frac{x}{6}$
8. $\frac{x}{3} - \frac{1}{2} < \frac{1}{6}$
9. $x - \frac{1}{3} = -(x - \frac{1}{3})$
10. $2(x - 3) = 2(x - 4)$
11. $|3 - 2x| = 19$
12. $3 - |2 + 7t| = -10$
13. $8m + 30m - 2m - 10 = 2m + 12 + 36m - 12$
14. $4x - (12 - 3x) = -5x + 12$
15. $2(2z - 3.1) + (z - 1) - 3.3 = 0$
16. $14(b - 3) + b - 12(5b + 1) = -72b$
17. $3(\frac{7}{9}r + 2) - 4r + 15 = 32 - 7r - 5\frac{2}{3}$

(continued on next page)

18. $100(p + 4) + 10p + p = 111p + 400$

19. $100(p + 3) + 10p + p = 111p + 400$

20. $-7(2a - 3) + 3(\frac{1}{3} - a) = 14 + 2(4 - 4379a)$

21. $13(s + 2) - 5(2s - 10) = -3(s + 4) + 10$

22. $x(x - 3) + x^2 = 2(x^2 - 6)$

23. $17(x - 6) \leq -5 - 16(x + 4)$

24. $60x^2 - 10x + 200 = 60x^2 + 6x + 203$

25. $x(x + 11) = 2(x^2 - 6)$

26. $2a^2 + 2(6a - 3) + 6(2a - 1) - a(2a + 3) = -5$

27. $14c - 1 + 4(-c - 2) + c^2 = 12c(c + 1) - 11c^2$

28. $33 - n^2 + 3n = 3 + 4(3n + 2)$

29. $9y(y + 1) - 2(y^2 - \frac{1}{2}) = y$

30. $(6x + 25)(6x - 25) = -49$

31. $(2x - 3)(2x - 3) + 2(2x - 3) + 1 = 0$

32. $\frac{1 - k}{12} + \frac{k}{5} - \frac{k - 1}{5} = \frac{1}{12}$

33. $\frac{y}{y - 1} + 2 = \frac{3}{5}$

34. $\frac{2}{3}(j + 1) - \frac{j - 8}{2} + 1 = \frac{1}{2}$

35. $\frac{700 + 300x + 10{,}000\% \text{ of } x}{15} = 100$

36. $1 - \frac{1}{d} = \frac{6}{d^2}$

37. $\frac{1}{2a} - \frac{a}{3} = \frac{-5}{12}$

38. $\frac{179}{2t - 13} = 1 - \frac{2t}{2t - 13}$

39. $4 - \frac{7}{z + 2} = \frac{z + 9}{z + 2} + z - 4$

40. $y(y - 6) = -5 - 16(y - \frac{29}{16})$

41. $\frac{37}{t} - 2 = \frac{37}{\frac{2}{3}t} - 1$

42. $\frac{3x + 1}{4x + 1} = \frac{6x}{8x - 1}$

43. $\frac{p + 2}{p - 3} = \frac{p + 4}{p - 1}$

44. $\frac{x - 2}{x - 3} - 1 = \frac{3}{x + 3}$

45. $\frac{3}{4}(\frac{2}{1 - h}) + \frac{1}{2} = \frac{24h - 31}{4(1 - h)}$

46. $\frac{1}{5}(r - 1) - \frac{1}{4}(r - 1) + \frac{1}{3}(r - 1) - \frac{1}{2}(r - 1) + (r - 1) = 0$

47. $43(r + 1)^2 - 52(2r + 2)^2 = 0$

48. $\frac{r + 1}{r - 2} - \frac{r + 3}{r - 4} = \frac{-6}{6r - r^2 - 8}$

M. Simplify.

1. $2x + 5x + 3 + 1$

2. $4z - 3z + 6 - 5z - 7$

3. $2a - 5 + 3a - 2 - 1$

4. $2(3a - 2) + 3(a - 2)$

5. $4(2d - 1) - (d - 2) + (d - 4)$

6. $-[(2x - 3) - (4x - 1)] - 2(3 - 2x)$

7. $6y - -3 + -2y - 4$

8. $4a - 2b + 3a - 6b - 2b$

9. $x^2 + 3x - x^4 - 4x$

10. $-2(a - 3) + 3(a + b) - 2b$

11. $x(x - 3) + x^2(3 - x) + 3(x^2 - x)$

12. $(x - 2)(x + 3) + 2(x - 4)$

13. $-2x(x^2y) - 3x^3y^2(-xy)$

14. $-3ab(-2a^3b^2 + 4ab^3)$

15. $-(2x - 3)(x + 2) - (x - 4)^2$

16. $3x^2[(x - 3)(x - 4) + (4 - x)(x - 3)]$

17. $\frac{-24a}{2}$

18. $\frac{6a^3b^2}{2a^2b}$

19. $\frac{6a^3b^2 - 4ab^2}{2ab^2}$

20. $\frac{2x^2}{3y} \cdot \frac{9y^2}{4x}$

21. $\frac{3x^2y^3}{4xy^4} \cdot \frac{24x^3y^2}{9x^4y}$

22. $\frac{20x^2y}{-6xy^3z} \div \frac{10xy^4}{3xz}$

23. $18x(\frac{2}{9x} + \frac{1}{6})$

24. $24x^2y(\frac{3y}{8x} - \frac{2x + 1}{12x^2} + \frac{1}{6x})$

25. $\frac{2}{3x} + \frac{3}{4x}$

26. $\frac{2}{x - 3} + \frac{3}{x - 4}$

(continued on next page)

27. $\dfrac{a - 1}{a + 4} - 2 + \dfrac{a}{2a + 4}$

28. $\dfrac{1 - \dfrac{x}{y}}{1 + \dfrac{x}{y}}$

29. $\dfrac{\dfrac{2}{3k} + \dfrac{1}{k}}{\dfrac{1}{2k} - \dfrac{1}{6}}$

30. $\dfrac{1 - \dfrac{1}{c - d} - \dfrac{1}{d - c}}{\dfrac{c + d}{c - d}}$

31. $\sqrt{(n + 2)^4(s - 1)^2}$

32. $\sqrt{4c^2 + 12cd + 9d^2}$

33. $\sqrt{2a^2}\,(\sqrt{18a^2} - \sqrt{32a^2})$

34. $\sqrt{2}\,(\sqrt{26} - \sqrt{22}) + \sqrt{3}\,(\sqrt{33} - \sqrt{39})$

N. In these exercises, write in the blanks the principles and generalizations (not just their names) which justify the steps in the proof.

1. For each x, for each y, $(-x)y + xy = 0$.

$(-x)y + xy$ ____________________

$= y(-x) + yx$ ____________________

$= y(-x + x)$ ____________________

$= y(x + -x)$ ____________________

$= y \cdot 0$ ____________________

$= 0.$

2. For each a, for each b, $2a + 3b - 2a = 3b$.

$2a + 3b - 2a$ ____________________

$= 2a + 3b + -(2a)$ ____________________

$= 3b + 2a + -(2a)$ ____________________

$= 3b + [2a + -(2a)]$ ____________________

$= 3b + 0$ ____________________

$= 3b.$

27. $-\frac{a^2 + 18a + 36}{2(a+4)(a+2)}$, $[-4 \neq a \neq -2]$

28. $\frac{y - x}{y + x}$, $[0 \neq y \neq -x]$

29. $\frac{10}{3 - k}$, $[0 \neq k \neq 3]$

30. $\frac{c - d}{c + d}$, $[c^2 \neq d^2]$

31. $|(n + 2)^2(s - 1)|$

32. $|2c + 3d|$

33. $-2a^2$

34. $\sqrt{11} - \sqrt{13}$

N. [In writing principles and generalizations for these exercises, students may interchange the members of the equations which follow the quantifiers. Hence, their answers may differ in this respect from those given here.]

1. $\forall_x \forall_y \; xy = yx$

   $\forall_x \forall_y \forall_z \; x(y + z) = xy + xz$

   $\forall_x \forall_y \; x + y = y + x$

   $\forall_x \; x + -x = 0$

   $\forall_x \; x \cdot 0 = 0$

2. $\forall_x \forall_y \; x - y = x + -y$

   $\forall_x \forall_y \; x + y = y + x$

   $\forall_x \forall_y \forall_z \; x + y + z = x + (y + z)$

   $\forall_x \; x + -x = 0$

   $\forall_x \; x + 0 = x$

3. $\forall_x \forall_{y \neq 0} \; -\frac{x}{y} = \frac{x}{-y}$

   $\forall_x \forall_y \; x - y = -(y - x)$

4. $\forall_x \forall_y \; x - y = x + -y$

   $\forall_x \forall_y \; -x + -y = -(x + y)$

   $\forall_x \; x = --x$

   $\forall_x \forall_y \forall_z \; x + y + z = x + (y + z)$

   $\forall_x \forall_y \; x - y = x + -y$

5. $\forall_x \forall_y \; x - y = x + -y$

   $\forall_x \forall_y \forall_z \; x + y + z = x + (y + z)$

   $\forall_x \forall_y \; -xy = -(xy)$

   $\forall_x \forall_y \forall_z \; (x + y)z = xz + yz$

   $-2 + -5 = -7$

   $\forall_x \forall_y \; -xy = -(xy)$

   $\forall_x \forall_y \; x - y = x + -y$

[4-104]

3. For each r, for each $s \neq r$, $\frac{1}{r - s} = -\frac{1}{s - r}$.

$$
\begin{aligned}
& -\frac{1}{s - r} \\
= \; & \frac{1}{-(s - r)} \\
= \; & \frac{1}{r - s} .
\end{aligned}
$$

4. For each a, for each b, for each c, for each d,
$a - b - (c - d) = a - b - c + d$.

$$
\begin{aligned}
& a - b - (c - d) \\
= \; & a - b + -(c + -d) \\
= \; & a - b + (-c + - -d) \\
= \; & a - b + (-c + d) \\
= \; & a - b + -c + d \\
= \; & a - b - c + d .
\end{aligned}
$$

5. For each r, for each s, for each t, $7r - 2s - 5s = 7r - 7s$.

$$
\begin{aligned}
& 7r - 2s - 5s \\
= \; & 7r + -(2s) + -(5s) \\
= \; & 7r + [-(2s) + -(5s)] \\
= \; & 7r + [(-2)s + (-5)s] \\
= \; & 7r + (-2 + -5)s \\
= \; & 7r + -7s \\
= \; & 7r + -(7s) \\
= \; & 7r - 7s .
\end{aligned}
$$

O. Solve these problems.

1. Mr. Adams paid \$127.00 for a plane ticket. This price included 10% Federal tax. How much was the tax?

2. Two numbers differ by 58.5. The smaller number is 55% of the larger. What are the two numbers?

3. The smallest of seven consecutive integers is $\frac{6}{7}$ of the largest. What are the seven integers?

4. Gundo owns goondols and Rassler owns ramlers. In all they own 48 goondols and ramlers. The number of ramlers that Rassler owns is $\frac{9}{7}$ the number of goondols that Gundo owns. How many ramlers does Rassler own?

5. In one of the redwood forests a certain tree is calculated to be twice as old as another; 150 years ago the first tree would have been 200 years older than the second tree was then. Find the present ages of the two trees.

6. Alphatown and Zilchville are 76 miles apart. At 8:00 a.m. on a certain day Zeke left Zilchville on his bicycle and started toward Alphatown. He traveled at the rate of 12 miles per hour, but rested 2 hours on the way. On the same day, Alex, who lives in Alphatown, left there at 9:00 a.m. on his bicycle and traveled toward Zilchville at a rate of 10 miles per hour all the way, with no stops to rest. How far from Zilchville were the boys when they met?

7. Charlotte went to the candy store and spent 57 cents for candy bars, bubble gum, and fruit drops. The candy bars were 6 cents each; the bubble gum was priced at two pieces for 1 cent; each package of fruit drops sold for 4 cents. Charlotte counted the number of candy bars, pieces of bubble gum, and packages of fruit drops and found that she had 57 in all. How many of each kind had she purchased?

O. 1. \$11.55 [$10x + x = 127$; the root is 11.55+]

2. 71.5 and 130 [$x - 0.55x = 58.5$]

3. 36, 37, ..., 42 [$x = (6/7)(x + 6)$]

4. 27 [x goondols, $(9/7)x$ ramlers; $x + (9/7)x = 48$]

5. 200 years; 400 years [x years...present age of the younger tree; $2x - 150 = (x - 150) + 200$]

6. This is a problem in which the data are insufficient. It is interesting, however, to make certain assumptions and see what results one will get.

(a) Assume that Zeke traveled for awhile, rested for 2 hours, started cycling again, and then met Alex.

x...hours Alex traveled until he met Zeke

$(x + 1) - 2$...hours Zeke traveled until he met Alex

$$10x + 12(x - 1) = 76, \quad x = 4$$

Since Alex traveled 10 miles per hour, in 4 hours he would have traveled 40 miles from Alphatown. So, the boys would have been 36 miles from Zilchville when they met.

(b) Assume that Zeke kept traveling until he met Alex, and then stopped to rest for 2 hours.

x...hours Alex traveled until he met Zeke

$x + 1$...hours Zeke traveled until he met Alex

$$10x + 12(x + 1) = 76, \quad x = 32/11$$

In 32/11 hours, Alex would travel 320/11 miles from Alphatown. So, the boys must have been almost 47 miles from Zilchville when they met.

Here are other questions that one might raise in connection with this problem.

(1) Suppose Zeke had traveled for 2 hours, and then stopped to rest. Would Alex have met him during this rest period? [Answer: No.]

(2) Suppose Zeke had traveled for 3 hours, and then stopped to rest for 2 hours. Would Alex have met him before he started traveling again? [Answer: Alex would have met him just at the end of his rest period, probably as he was getting on his bike to take off!]

O. S

(3) Suppose Zeke had traveled for 2 hours, rested for 1 hour, and then started cycling again. How far apart would the boys be at noon? [Answer: 10 miles.]

7. This problem does not yield to a straight-forward equation procedure. Suppose Charlotte has x candy bars and y pieces of bubble gum. Then, she has 57 - x - y packages of fruit drops. So, we are looking for whole numbers x and y of arithmetic such that

$$(1) \qquad 6x + \frac{1}{2}y + 4(57 - x - y) = 57.$$

Now, we could accumulate, by trial-and-error, many ordered pairs of whole numbers of arithmetic which satisfy (1), and determine which pairs fit the conditions of the problem. But, let's try a more systematic procedure. Solve (1) for 'x' to get:

$$(2) \qquad x = \frac{7y - 342}{4}.$$

Since x is a number of arithmetic, the values of 'y' in which we are interested are among those for which $7y - 342 > 0$, that is, those which are greater than 342/7. But, since y is a whole number of arithmetic less than 56, the values we want to consider are 49, 50, 51, 52, 53, 54, and 55. Since Charlotte spent at least 1 cent on bubble gum, the interesting values of 'y' are the even numbers 50, 52, and 54. Since x is a whole number and 4 is not a factor of 342, 4 cannot be a factor of y. So, we have left for consideration just the values 50 and 54 for 'y'. The corresponding values of 'x' are 2 and 9, respectively. But, 9 candy bars and 54 pieces of bubble gum would give Charlotte more than 57 items. So, it must be the case that Charlotte bought 2 candy bars, 50 pieces of bubble gum, and 5 packages of fruit drops.

*

Students who enjoy the type of problem in Exercise 7 will also enjoy these:

(1) A printer had a stock of pamphlets. He put a sixth of them on one table, several fifths on a second, and 4 on a third. How many pamphlets did he have in all?

(2) To number the pages of a bulky volume the printer used 1890 digits. How many pages has the volume? [Stanford University Mathematics Examination, April 1957]

[4-106]

O. S

's

4/158

(3) Bob's stamp collection consists of three books. One fifth of his stamps are in the first book, several sevenths in the second book, and 303 stamps in the third book. How many stamps has Bob? [Adapted from the Stanford University Competitive Examination in Mathematics, March 1957.]

Some excellent word problems are on pages 149-154 and pages 156-158 of Burton W. Jones' Elementary Concepts of Mathematics (New York: The Macmillan Company, 1947).

*

8. 4 feet by 7 feet [$x(x + 3) + 22 = (x + 1)\{(x + 3) + 3\}$]

9. The degree-measure of $\angle A$ is 39, that of $\angle B$ is 52, that of $\angle C$ is 89 [$3x + 4x + (7x - 2) = 180$]

10. 45¢ grade, 50; 39¢ grade, 100 [$45x + 39(600 - x) = 41 \cdot 600$]

11. $22400 [x...selling price, $500 + 0.025(x - 10000)$...real estate agent's commission; $x + \{500 + 0.025(x - 10000)\} = 23210$]

12. quarters, 7; dimes, 13; nickels, 35 [$25x + 10(2x - 1) + 5\{2(2x - 1) + 9\} = 480$]

13. 2.4 days [$(1/5) + (x/3) = 1$]

14. 85% alloy, 14; 55% alloy, 6 [$0.85x + 0.55(20 - x) = 0.76(20)$]

15. 180 [$\frac{1}{4}\{\frac{4}{5}(\frac{5}{6}n - 25)\} = 25$]

16. 4/13 gallons [$0.85(1) = 0.65(1 + x)$]

8. Mrs. Beccardi has a rectangular flower garden which is 3 feet longer than it is wide. She wants to plant a new variety of tulips; so, she decides to enlarge the garden by making it 1 foot wider and 3 feet longer. This, of course, gives more area in the garden plot. Mrs. Beccardi calculates that she will have 22 square feet more in her garden after she makes the changes. What are the dimensions of the garden before Mrs. Beccardi enlarges it?

9. Suppose the sum of the degree-measures of the angles of a triangle is 180. The measures of angle A and angle B are in the ratio 3:4. The degree-measure of angle C is 2 less than the sum of the degree-measures of angles A and B. Find the degree-measure of each angle.

10. A dealer wishes to prepare 150 gallons of an oil blend to sell at 41 cents a quart. He plans to mix oil selling for 45 cents a quart with a cheaper grade that sells for 39 cents a quart. How many gallons of each kind should he use?

11. Mr. Jones received a check for $23,210 for the sale of a house. The buyer included the real estate agent's commission when he wrote the check. This commission was computed at a rate of 5% on the first ten thousand dollars [of the selling price] and $2\frac{1}{2}$% on the remainder. After giving the real estate agent his commission, how much did Mr. Jones have left? That is, what was the selling price?

12. A money box contains nickels, dimes, and quarters. The number of dimes is 1 less than twice the number of quarters; the number of nickels is 9 more than twice the number of dimes. Altogether the coins are worth $4.80. How many coins of each kind are in the box?

13. David and his cousin Dick can do a certain piece of work in 3 days when they work together. David can do this job in 5 days when he works alone. He starts on it one morning when Dick has gone hunting, and works all day. The next day Dick joins him at the task, and works with him until the job is done. How many days did Dick work with David?

(continued on next page)

14. A jeweler who makes rings has two alloys of gold, one being 85% pure gold and the other 55% pure gold. How many ounces of each of the alloys must be taken to make 20 ounces of a new alloy that will be 76% pure gold?

15. A group of paratroopers are making their first practice jumps. One sixth of them jumped at the first drop zone; 25 jumped at the second; twenty per cent of the remainder left the plane at the third drop zone, and three-fourths of those remaining jumped at the fourth drop zone. There were 25 fellows remaining on the plane; they decided not to jump, even if this meant they would be transferred out of the division. How many men were in the original group?

16. A druggist has a gallon of an alcohol solution that is 85% pure alcohol, and wishes to reduce it to a solution which is 65% pure alcohol. How much distilled water must he add to the original solution to obtain the "65% pure" solution?

P. Find the approximation correct to the nearest .01.

1. $\sqrt{6}\cdot\sqrt{2}$
2. $\dfrac{\sqrt{250}}{\sqrt{2}}$
3. $\sqrt{32} - \sqrt{8}$
4. $\sqrt{27} + \sqrt{75}$
5. $\sqrt{32}\cdot\sqrt{14}\cdot\sqrt{21}$
6. $\sqrt{108} + \sqrt{48} - \sqrt{3}$

Q. Simplify. [Use scientific notation to express the result.]

1. $\sqrt{25 \cdot 10^8} + \sqrt{36 \cdot 10^6}$
2. $\dfrac{(.144)(14000)}{(.000012)(.006)}$
3. $\dfrac{14000(10^{24})}{.007(10^{-20})}$
4. $\dfrac{\sqrt{.00000009}\cdot(2000)^8}{(.0006)^4\cdot\sqrt{40000}}$
5. $\dfrac{.236(10^{16}) + 2.36(10^{18})}{.00236(10^{20})(10^5)}$

P. 1. 3.46 2. 11.18 3. 2.83 4. 13.86

5. If the student simplifies '$\sqrt{32} \cdot \sqrt{14} \cdot \sqrt{21}$' to '$56\sqrt{3}$' and uses 1.7321 as the approximation to $\sqrt{3}$ correct to the nearest 0.0001, he cannot be <u>sure</u> that he has found the approximation to $\sqrt{32} \cdot \sqrt{14} \cdot \sqrt{21}$ correct to the nearest 0.01.

Since 1.7321 <u>is</u> correct to the nearest 0.0001, its error is less than 0.00005. $56 \times .00005 = 0.0028$, so if one uses 1.7321 as the approximation to $\sqrt{3}$, the error in the approximation to $56\sqrt{3}$ will be less than 0.0028.

$$56 \times 1.7321 = 96.9976$$

So,

$$(*) \qquad 96.9948 < 56\sqrt{3} < 97.004.$$

Is 97 the approximation correct to the nearest 0.01? No, because to say this would mean that $96.995 < 56\sqrt{3} < 97.005$, and one cannot conclude this from (*). One <u>can</u> say that 97 is the approximation to $56\sqrt{3}$ correct to the nearest 0.1, for he can conclude from (*) that $96.95 < 56\sqrt{3} < 97.05$.

[Even using 1.73205 as an approximation to $\sqrt{3}$ will not enable one to determine an approximation to $56\sqrt{3}$ correct to the nearest 0.01. Using 1.73205, one finds that $96.99452 < 56\sqrt{3} < 96.99508$. [By more labor, one can find that $56\sqrt{3}$ <u>is</u> 96.99 correct to the nearest 0.01.]]

6. 15.59

Q. 1. $\sqrt{25 \cdot 10^8} + \sqrt{36 \cdot 10^6}$

$= 5 \cdot 10^4 + 6 \cdot 10^3$

$= 10^3 (5 \cdot 10 + 6)$

$= 10^3 \times 56$

$= 5.6 \times 10^4$

2. $\dfrac{(.144)(14000)}{(.000012)(.006)}$

$= \dfrac{1.44 \times 10^{-1} \times 1.4 \times 10^4}{1.2 \times 10^{-5} \times 6 \times 10^{-3}}$

$= \dfrac{2.016 \times 10^3}{7.2 \times 10^{-8}}$

$= .28 \times 10^{11}$

$= 2.8 \times 10^{10}$

[4-108]

4/160

3. $$\frac{14000(10^{24})}{.007(10^{-20})}$$

$$= \frac{1.4 \times 10^4 \times 10^{24}}{7 \times 10^{-3} \times 10^{-20}}$$

$$= \frac{1.4 \times 10^{28}}{7 \times 10^{-23}}$$

$$= .2 \times 10^{51}$$

$$= 2 \times 10^{50}$$

4. $$\frac{\sqrt{.00000009} \cdot (2000)^8}{(.0006)^4 \cdot \sqrt{40000}}$$

$$= \frac{.0003 \times 2^8 \times 10^{24}}{6^4 \times 10^{-16} \times 200}$$

$$= \frac{3 \times 10^{-4} \times 256 \times 10^{24}}{1296 \times 10^{-16} \times 2 \times 10^2}$$

$$= \frac{768 \times 10^{20}}{2592 \times 10^{-14}}$$

$$= .296296\ldots \times 10^{34}$$

$$= 2.96296\ldots \times 10^{33}$$

5. $$\frac{.236(10^{16}) + 2.36(10^{18})}{.00236(10^{20})(10^5)}$$

$$= \frac{2.36 \times 10^{15} + 2.36(10^{18})}{2.36(10^{-3})(10^{20})(10^5)}$$

$$= \frac{2.36(10^{15} + 10^{18})}{2.36(10^{22})}$$

$$= \frac{10^{18}(10^{-3} + 1)}{10^{22}}$$

$$= 10^{-4}(10^{-3} + 1)$$

$$= 10^{-7} + 10^{-4}$$

$$= .0000001 + .0001$$

$$= .0001001$$

$$= 1.001 \times 10^{-4}$$

<u>Alternative</u>

$$= \frac{10^{15} + 10^{18}}{10^{22}}$$

$$= 10^{-7} + 10^{-4}$$

$$= .0000001 + .0001$$

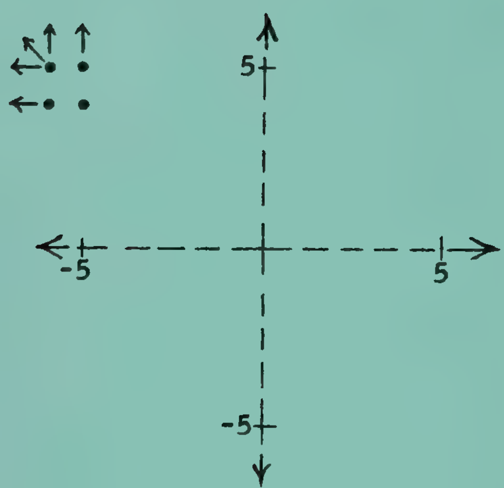
$$= .0001001$$

$$= 1.001 \times 10^{-4}$$

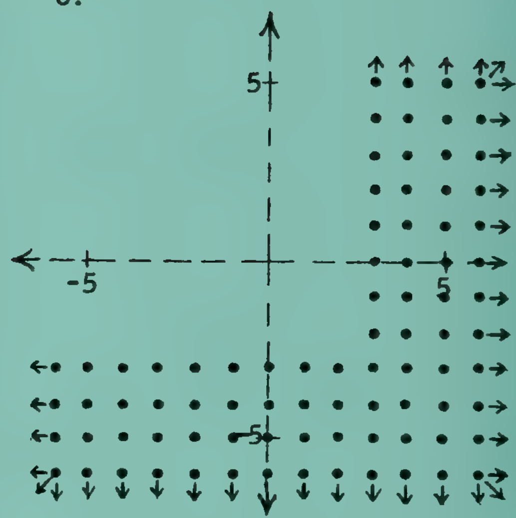
[4-108]

[4-109]

5.



6.



III. 1. The set of all ordered pairs of integers, such that the first component is −5 or the second component is 5 or 6.

2. The set of all ordered pairs of integers such that the first component is between 0 and −5 and the second component is between 2 and −2.

3. $\{(x, y),\ x \text{ and } y \text{ integers: } y = 2x - 1\}$

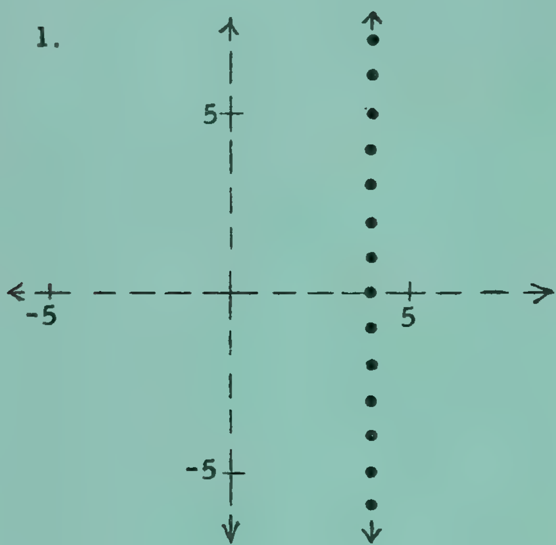
4. $\{(x, y),\ x \text{ and } y \text{ integers: } (x = -3 \text{ and } -5 \leq y \leq -1) \text{ or } (y = -3 \text{ and } -5 \leq x \leq -1)\}$

[4-109]

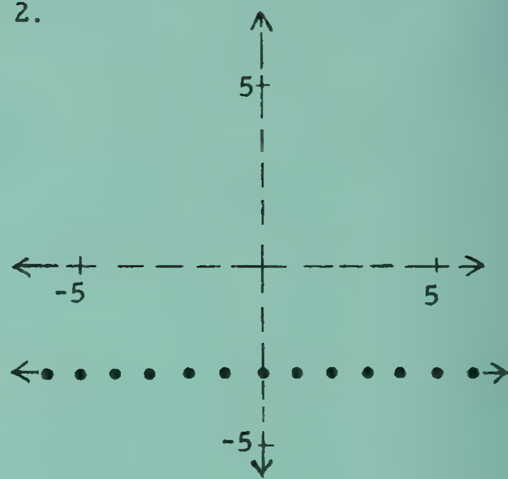
Answers for TEST.

I. 1. 35 2. 7 3. 7 4. 5 5. 7

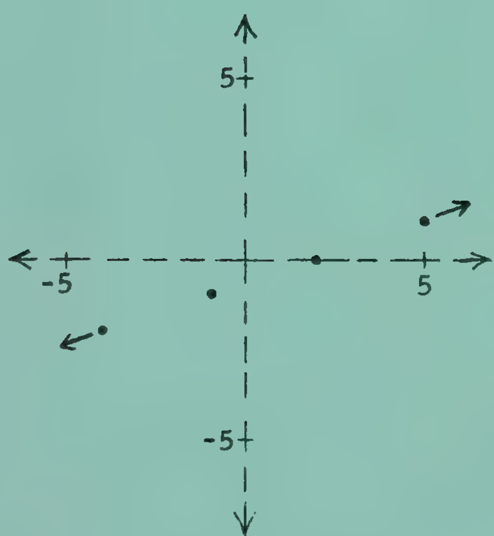
II. 1.



2.



3.



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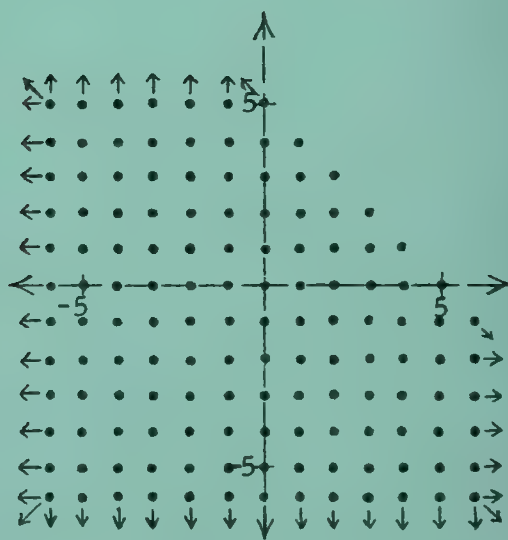


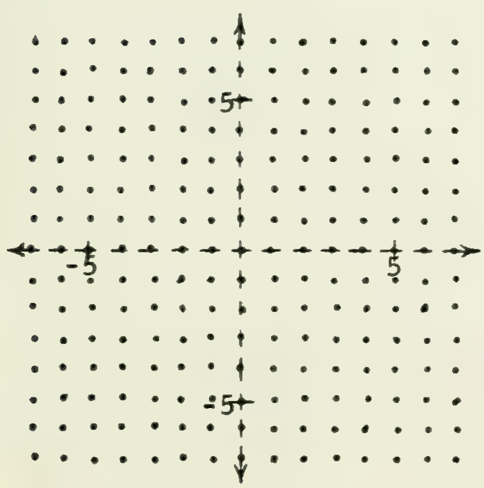
TEST

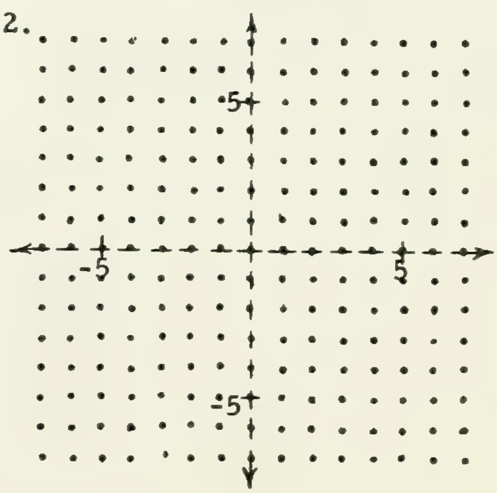
I. Suppose A = {2, 3, 5, 6, 7} and B = {2, 6, 15, 24, 35, 48, 63}.

1. How many ordered pairs of numbers are in $A \times B$?
2. How many ordered pairs of numbers in $A \times B$ will have first component 3?
3. How many ordered pairs of numbers in $A \times B$ will have first component 6?
4. How many ordered pairs of numbers in $A \times B$ will have second component 6?
5. If you listed the ordered pairs of numbers in $B \times A$, how many would have second component 6?

II. Use these pictures of the number plane lattice and plot the sets of points described below.

1. The set of all ordered pairs of integers which correspond with dots that have first coordinate 4.
2. The set of all ordered pairs of integers with second component -3.
3. $\{(x, y),\ x \text{ and } y \text{ integers: } x = 2 + 3y\}$
4. $\{(x, y),\ x \text{ and } y \text{ integers: } x + y \leq 5\}$
5. $\{(x, y),\ x \text{ and } y \text{ integers: } x < -4 \text{ and } y > 3\}$
6. $\{(x, y),\ x \text{ and } y \text{ integers: } x > 2 \text{ or } y < -2\}$

1.


2.


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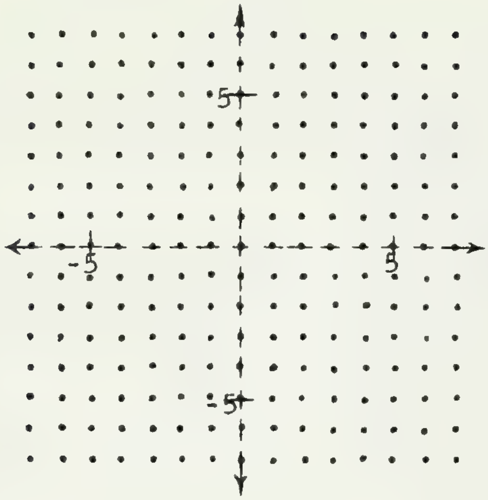
3.



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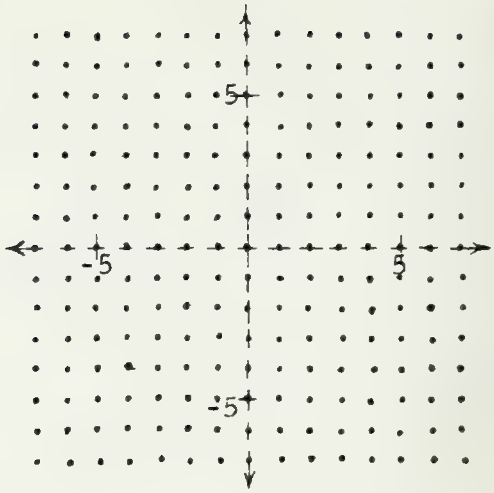
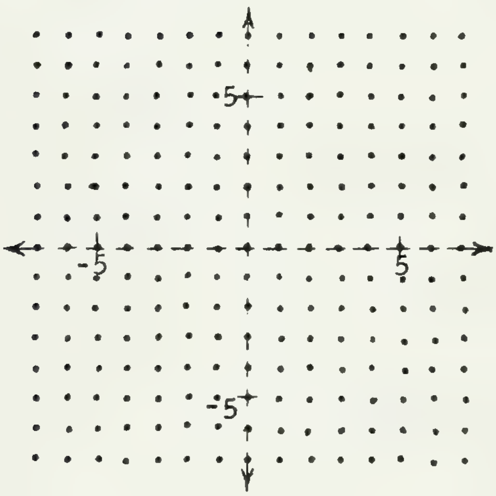
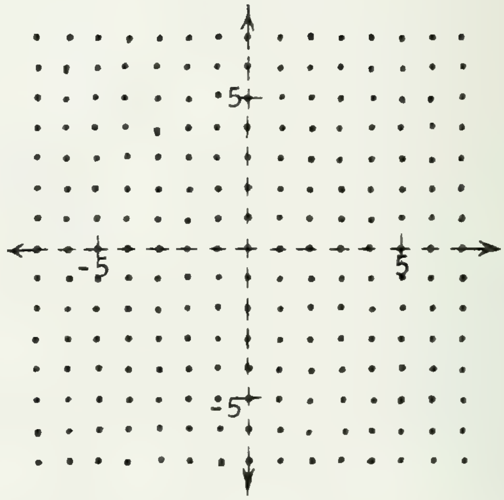


5.

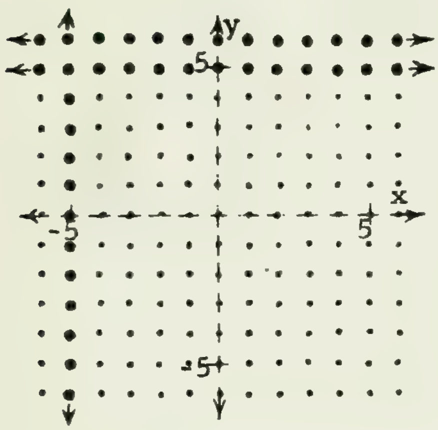


6.

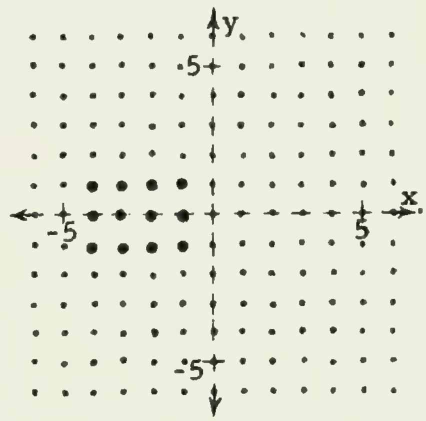


III. Here are pictures of sets of ordered pairs of integers. For the first two charts, write word-descriptions of the sets pictured. For the third and fourth charts, use brace-notation to describe the set pictured.

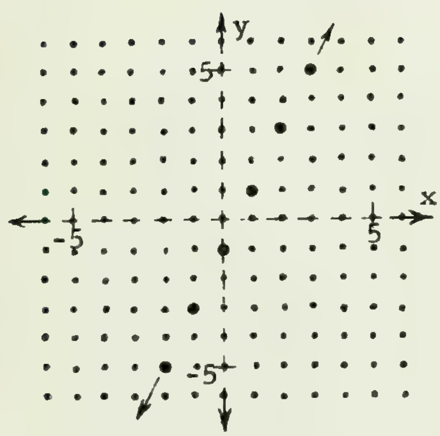
1.



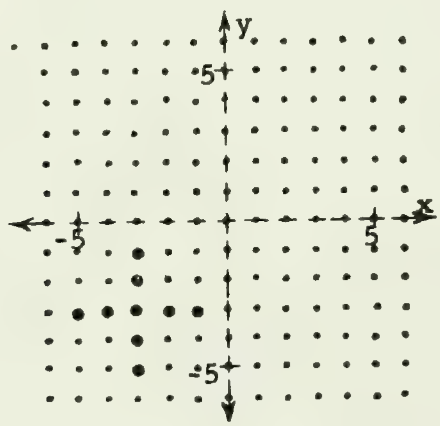
2.



3.



4.



IV. In each of the following exercises, you are given descriptions of a set P and a set Q. For each exercise,

(a) plot the points in each set on the same diagram,

(b) tell the number of points in each set,

(c) tell the number of points in the intersection, and

(d) tell the number of points in the union.

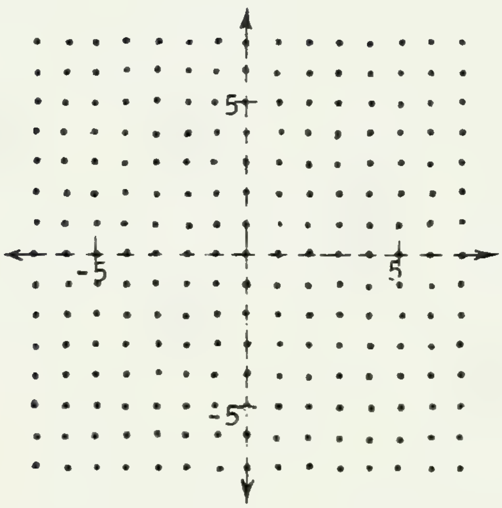
1. $P = \{(x, y),\ x \text{ and } y \text{ integers: } -5 < x < 1 \text{ and } 5 > y > 1\}$

   $Q = \{(x, y),\ x \text{ and } y \text{ integers: } |x| \leq 3 \text{ and } |y| \geq 4\}$

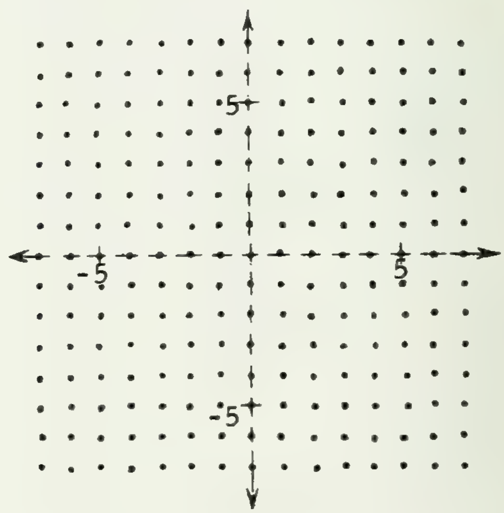
2. $P = \{(x, y),\ x \text{ and } y \text{ integers: } 6 > x > 3 \text{ and } 0 > y > -3\}$

   $Q = \{(x, y),\ x \text{ and } y \text{ integers: } 0 < x < 3 \text{ and } 3 < y < 6\}$

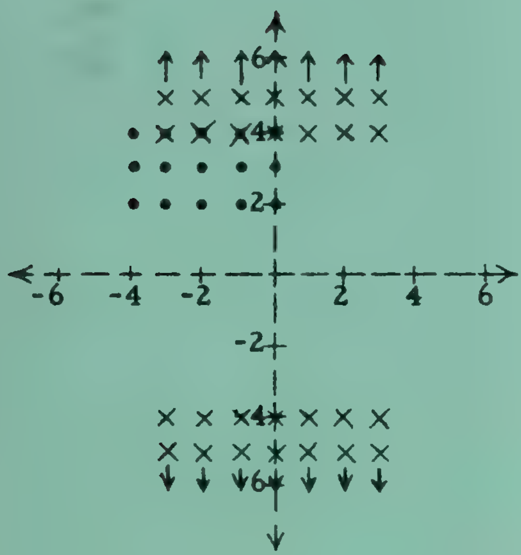
1.



2.



IV. 1. (a)

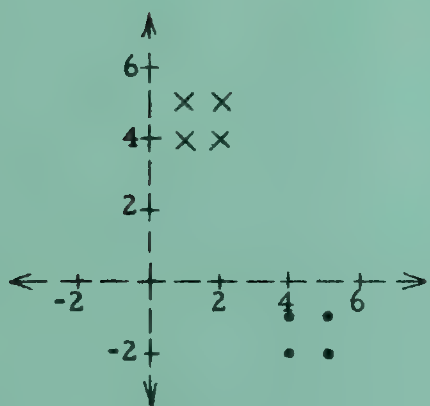


(b) n(P) = 15, n(Q) = ____ This question makes no sense because set Q is infinite.

(c) $n(P \cap Q) = 4$

(d) $n(P \cup Q) =$ ____ This question makes no sense because the set is infinite.

2. (a)



(b) n(P) = 4, n(Q) = 4

(c) $n(P \cap Q) = 0$

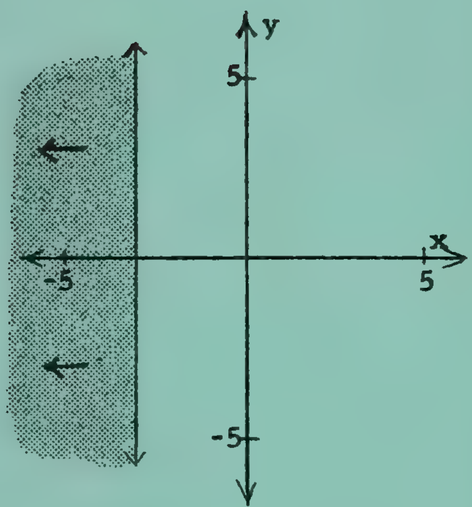
(d) $n(P \cup Q) = 8$

[4-112]

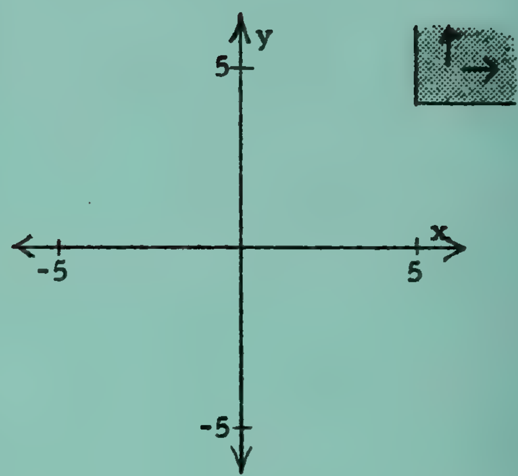
IV

[4-113]

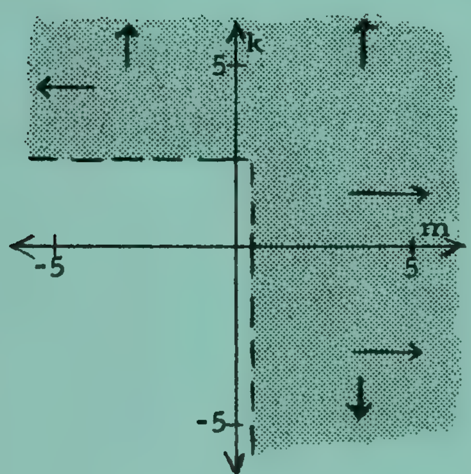
V. 1.



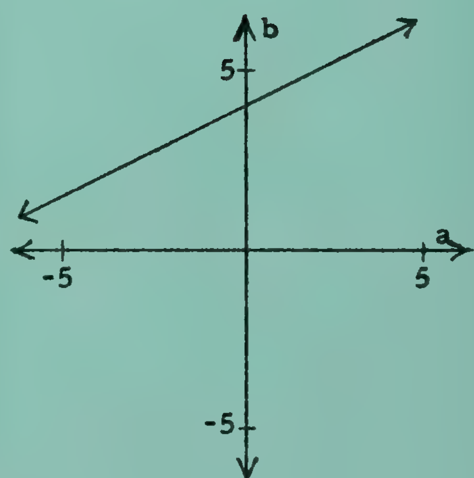
2.



3.

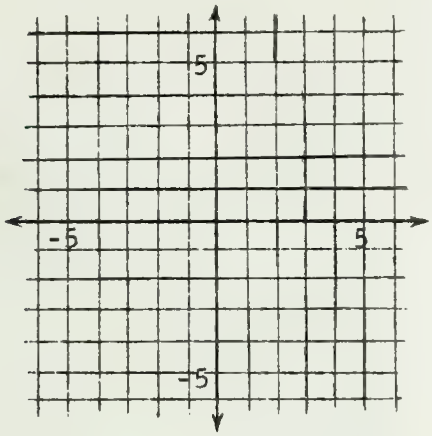


4.

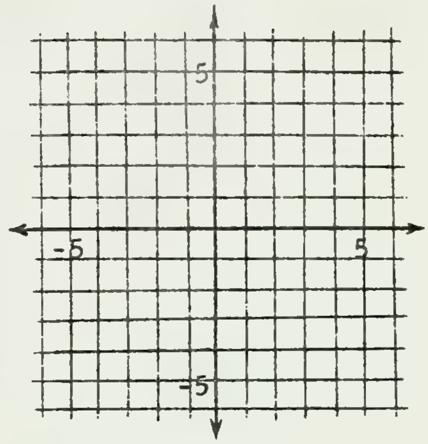


V. Graph these sets.

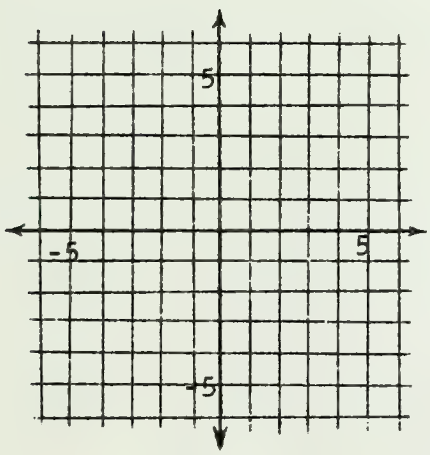
1. $\{(x, y):\ x \leq -3\}$



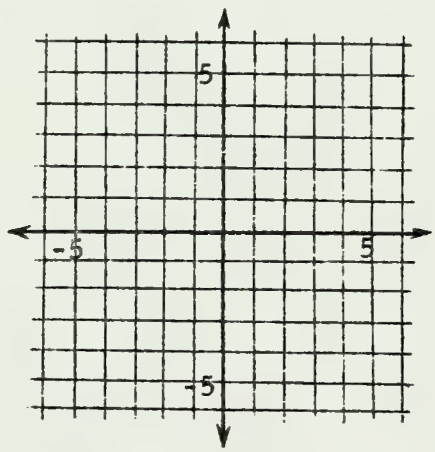
2. $\{(x, y):\ x \geq 5 \text{ and } y \geq 4\}$



3. $\{(m, k):\ m > \frac{1}{2} \text{ or } k > 2\frac{1}{2}\}$



4. $\{(a, b):\ b = \frac{1}{2}a + 4\}$



VI. Six sets (straight lines) are pictured below.

1. List the points which belong to the sets.

(1) $a \cap e$ (2) $a \cap d$

(3) $c \cap e$ (4) $b \cap a$

(5) $d \cap (a \cup b)$

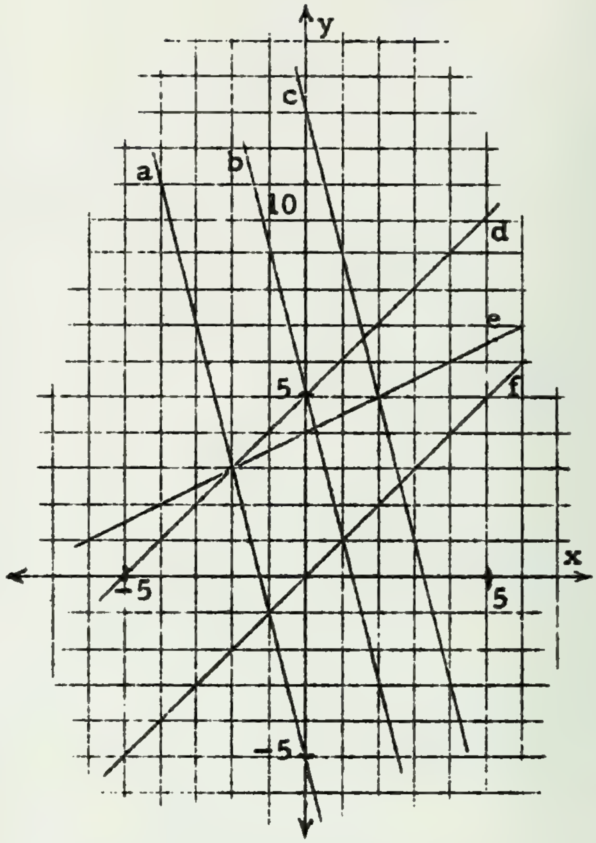
(6) $f \cap b$

(7) $f \cap$ (the x-axis)

(8) $d \cap$ (the y-axis)

(9) $(f \cap a) \cup (f \cap b)$

(10) $e \cap (a \cup c)$



2. In the blank at the left of each of the following equations, write the letter which is the name of its locus.

_________ (1) $y = x$ _________ (2) $y = -4x - 5$

_________ (3) $y = -4x + 5$ _________ (4) $y = \frac{1}{2}x + 4$

_________ (5) $y = x + 5$ _________ (6) $y = -4x + 13$

3. Use the picture to tell which ordered pairs satisfy both equations.

(1) $\left.\begin{array}{l} y = -4x - 5 \\ y = -4x + 13 \end{array}\right\}$ (2) $\left.\begin{array}{l} y = x \\ y = -4x + 5 \end{array}\right\}$

(3) $\left.\begin{array}{l} y = -4x - 5 \\ y = \frac{1}{2}x + 4 \end{array}\right\}$ (4) $\left.\begin{array}{l} y = -4x + 5 \\ y = x + 5 \end{array}\right\}$

VI. 1. (1) (-2, 3) (2) (-2, 3) (3) (2, 5)

(4) These lines appear to be parallel, so $b \cap a = \emptyset$.

(5) (-2, 3) and (0, 5) (6) (1, 1) (7) (0, 0)

(8) (0, 5) (9) (-1, -1) and (1, 1) (10) (-2, 3) and (2, 5)

2. (1) f (2) a (3) b (4) e (5) d (6) c

3. (1) No ordered pairs satisfy both equations since the lines are parallel.

(2) (1, 1) (3) (-2, 3) (4) (0, 5)

VII. 1. $A = h_1\ell + h_2\ell + 2[h_1w - \frac{1}{2}(h_1 - h_2)w]$

$A = \ell(h_1 + h_2) + w(h_1 + h_2)$

$= (\ell + w)(h_1 + h_2).$

[If your students know the formula for the area-measure of a trapezoid, they may submit the following in answer to Exercise 1.

$$A = h_1\ell + h_2\ell + 2\left[\frac{w}{2}(h_1 + h_2)\right]$$
$$= \ell(h_1 + h_2) + w(h_1 + h_2)$$
$$= (\ell + w)(h_1 + h_2).$$

This is acceptable, of course!]

2. $A = (20 + 8)(16 + 12)$

$= 784.$

[4-114]

VI.

[4-115]

VI.

VIII. 1. positive integers, 14, 14

2. integers, 9, 9

3. rationals, $-\frac{31}{2}$, $-\frac{31}{2}$

4. rationals, $\frac{9}{10}$, $\frac{9}{10}$

5. 5, positive integers, 5, 5

IX. 1. 5, 13, 19, 31

2. $1, 2, 2^2, 3, 3^2, 2 \cdot 3, 2^2 \cdot 3, 2 \cdot 3^2, 2^2 \cdot 3^2$

3. $2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$; $2 \cdot 3 \cdot 5 \cdot 5$

4. 1, 2, 3, 5, 6, 10, 15, 25, 30, 50, 75, 150

5. $2^3 \cdot 3^2 \cdot 5^2$

X. 1. (a) r^4s^2 (b) $(d^2 + e)(d + e^2)$ [or: $d^3 + ed + d^2e^2 + e^3$]

(c) $7^2 + n^3$ (d) $\frac{3^3a^4}{b^3}$ (e) $\frac{5^2n^2 + 7^3k^2}{3^3a^3 - 2^2b^2}$

2. (a) n^{12} (b) 4^6 [or: 4096] (c) 2^7x^8 [or: $128x^8$]

(d) $-6^2c^4d^3$ [or: $-36c^4d^3$] (e) $-\frac{n^2}{3p^2}$, [$np \neq 0$]

(f) $2^3 \cdot 3^2a^7b^8$ [or: $72a^7b^8$]

(g) $-\frac{1}{2^3}e^9f^3g^6$ [or: $-\frac{1}{8}e^9f^3g^6$]

(h) $\frac{1}{28x^2y}$, [$xy \neq 0$] (i) $\frac{2^3a^{15}b^6}{3^3c^3}$ [or: $\frac{8a^{15}b^6}{27c^3}$], [$ac \neq 0$]

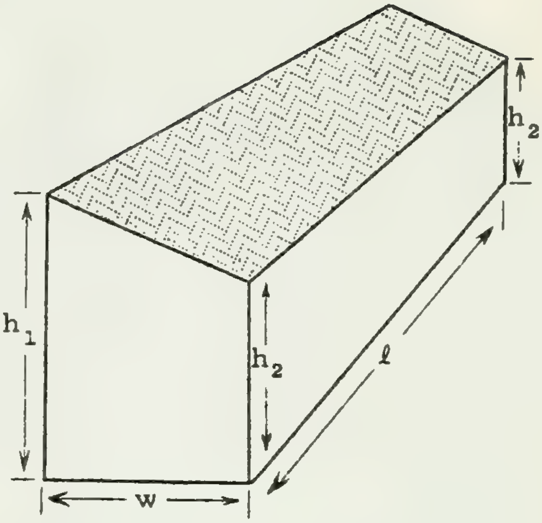
(j) $\frac{4^2x^4}{y^4}$ [or: $\frac{16x^4}{y^4}$], [$y \neq 0$]

3. (a) $(a - 2)(a + 2)(a^2 + 4)$ (b) $(6 - c)^2$

(c) $5ab(a^2 - 5)$ (d) $(5n + 2)(2n - 3)$

(e) $2(7 - d)(3 + 2d)$

VII. Here is a picture of a granary seen on an Iowa farm. In order to get an estimate on the cost of painting the exterior walls (not the roof) the farmer needs to know how many square feet of wall surface is to be painted.



1. Write a formula which would enable the farmer to determine the area measure of the exterior walls.

2. Suppose that the granary is 20 feet long and 8 feet wide, and that the longer and shorter heights are 16 feet and 12 feet respectively. How many square feet of exterior wall surface would there be?

VIII. Complete each of these sentences to true ones in at least one way.

1. 4 is a factor of 56 with respect to the set of __________ because 4, 56, and _____ belong to this set and $56 = 4 \cdot$ _____.

2. -8 is a factor of -72 with respect to the set of __________ because -8, -72, and _____ belong to this set and $-72 = -8 \cdot$ _____.

3. -6 is a factor of 93 with respect to the set of __________ because -6, 93, and _____ belong to this set and $93 = -6 \cdot$ _____.

4. $\frac{2}{3}$ is a factor of $\frac{3}{5}$ with respect to the set of __________ because $\frac{2}{3}$, $\frac{3}{5}$, and _____ belong to this set and $\frac{3}{5} = \frac{2}{3} \cdot$ _____.

5. _____^{2} is a factor of $35 \cdot 50$ with respect to the set of __________ because _____^{2}, $35 \cdot 50$, and 70 belong to this set and $35 \cdot 50 =$ _____$^2 \cdot 70$.

IX. 1. Which of these numbers are prime numbers?

5, 8, 13, 19, 21, 31, 42, 57682

2. List all numbers which are factors of the composite number 36.

3. Give the prime factorization for the number 72; for the number 150.

4. Use the prime factorization you found in Exercise 3 in making a list of all the factors of the number 150.

5. Give the prime power factorization of the number 1800.

X. 1. Use exponents to simplify each expression.

(a) $(rs)(rs)(rr)$ (b) $(dd + e)(d + ee)$ (c) $7 \cdot 7 + nnn$

(d) $\frac{3 \cdot 3 \cdot 3aaaa}{bbb}$ (e) $\frac{5 \cdot 5nn + 7 \cdot 7 \cdot 7kk}{3 \cdot 3 \cdot 3aaa - 2 \cdot 2bb}$

2. Simplify.

(a) $n^5 \cdot n^7$ (b) $4^2 \cdot 4^4$ (c) $2^3 \cdot 2^4 \cdot x^3 \cdot x^5$

(d) $(6c^3d)(-6cd^2)$ (e) $\frac{-14n^3p^5}{42np^7}$

(f) $(2ab^2)^3(3a^2b)^2$ (g) $(-\frac{1}{2}e^3fg^2)^3$

(h) $\frac{28^2xy^4}{28^3x^3y^5}$ (i) $\left(\frac{2a^6b^2}{3ac}\right)^3$

(j) $\left(\frac{4x^2}{3y^3}\right)^2 \cdot (3y)^2$

3. Factor each of these pronumeral expressions.

(a) $a^4 - 16$ (b) $36 - 12c + c^2$ (c) $5a^3b - 25ab$

(d) $10n^2 - 11n - 6$ (e) $42 + 22d - 4d^2$

[4-117]

IX.)

4. (a) $6y$ (b) hj^2k^2 (c) $3 + n$

   (d) $r - s$ (e) 1

5. (a) $12u^3v^3$ (b) $30a^2b^2$ (c) $625 - n^4$

   (d) $(\frac{1}{2} - n)^2(\frac{1}{2} + n)$ (e) $(3 + c)(2 - 3c)(c - 5)$

XI. 1. $\frac{3a^3b + 10}{5a^2b^2}$, $[ab \neq 0]$ 2. $\frac{-r^2 - 4r - 6}{r^2 + 6r + 9}$, $[r \neq -3]$

3. 0, $[c \neq -4]$ 4. $\frac{121y - 152}{60y^2 - 240}$, $[-2 \neq y \neq 2]$

XII. 1. 1.792×10^3 2. 5.843×10^2 3. 3.4567×10^6

4. 3.69×10^2 5. 4.68×10^2 6. 5.79×10^2

7. 6.79×10^{-2} 8. 3.45×10^{-3} 9. 7.654×10^4

XIII. 1. 2.304×10^6 2. 2.85×10^2 3. 1.21×10^{-2}

4. 1.2×10 5. 1.25×10^{-2}

4. Find an HCF of the given pronumeral expressions.

(a) $12y,\ 18y^2$ (b) $7h^3j^2k^4,\ 6hj^2k^2$

(c) $3 + n,\ 9 - n^2$ (d) $r^2 - 2rs + s^2,\ r^2 - s^2$

(e) $a^2 - ab - 2b^2,\ 3a^2 - 2ab - 15b^2$

5. Find an LCM of the given pronumeral expressions.

(a) $12u^2v,\ 2uv^2,\ 6u^3v^3$ (b) $5ab^2,\ 2ab,\ 3a^2b$

(c) $625 - n^4,\ n^2 + 25,\ n - 5$ (d) $\frac{1}{4} - n + n^2,\ .25 - n^2$

(e) $3 + c,\ 2 - 3c,\ c - 5$

XI. Simplify.

1. $\frac{3a}{5b} + \frac{6}{3a^2b^2}$ 2. $\frac{r}{r^2 + 6r + 9} - \frac{r + 2}{r + 3}$

3. $\frac{9c - 21}{3c + 12} + \frac{7 - 3c}{4 + c}$ 4. $\frac{2}{5y - 10} + \frac{5}{3y + 6} - \frac{y}{20y^2 - 80}$

XII. For each number listed below, write its name in scientific notation.

1. 1792 2. 584.3 3. 3,456,700

4. 0.369×10^3 5. 0.0468×10^4 6. 0.00579×10^5

7. 0.000679×10^2 8. 0.00345 9. $7,654,000 \times 10^{-2}$

XIII. Simplify, and use scientific notation to express the results.

1. $480,000 \times 3200 \times .0015$ 2. $(57 \times 10^3) \times 0.005$

3. $(11 \times 10^{-2})^2$ 4. $(12^2 \times 10^3)^2 \times (\frac{1}{12} \times 10^{-2})^3$

5. $\frac{(8 \times 10^{-3}) \times (45 \times 10^2)}{(9 \times 10^{-6}) \times (32 \times 10^7)}$

SUPPLEMENTARY EXERCISES

A. Consider the cartesian product

$$\{-3, -2, -1, 0, 1, 2, 3\} \times \{-5, -4, -3, -2, -1, 0, 1, 2, 3\}.$$

Make a picture of this lattice.

Tell how many dots are graphs of ordered pairs with

1. first component -2.
2. second component 3.
3. first number less than or equal to -2.
4. second component greater than or equal to 0.
5. first number 4.
6. first number less than -1 and second number greater than -1.
7. first number less than -1 or second number greater than -1.
8. first number 2 less than second number.
9. second component 3 more than first component.
10. second number equal to 2 less than 3 times first number.
11. second component 2 less than 3 times first component and with second component twice the first component.

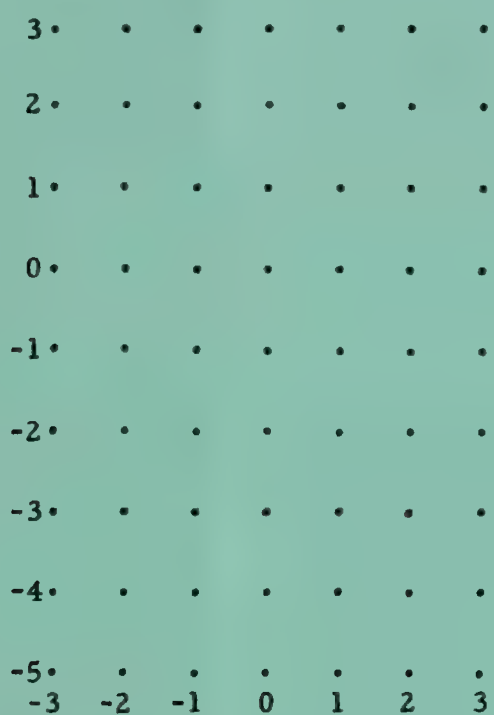
B. Draw a picture of the number plane lattice [your diagram should contain enough dots so that you can plot the points $(-6, 6)$, $(-6, -6)$, $(6, -6)$, and $(6, 6)$.] Plot the sets of points described below. Mark the dots in some particular fashion so that you can tell the sets apart. [We abbreviate 'real integers' to 'integers'.]

The set of all ordered pairs of integers such that...

1. first component is 1 more than second component.
2. first component is 2 less than second component.
3. the sum of the components of each ordered pair is 7.
4. 6 is the sum of the first component and twice the second component.

Answers for SUPPLEMENTARY EXERCISES.

A. Here is a picture of the lattice.



1. 9   2. 7   3. 18   4. 28   5. 0

6. 8 [The dots whose coordinates satisfy the conditions in question 6 are the graphs of (−2, 0), (−2, 1), (−2, 2), (−2, 3), (−3, 0), (−3, 1), (−3, 2), (−3, 3).]

7. 38 [The dots whose coordinates satisfy the requirement in question 7 are all the dots in the (−2)-column, the (−3)-column, the (0)-row, the (1)-row, the (2)-row, and the (3)-row. There are 18 dots in the two columns; the dots in the rows total 28. But, the 4 rows include 8 of the dots in the two columns. So, the total number of dots which satisfy the requirement of the problem is 18 + 28 - 8, or 38.]

A. C

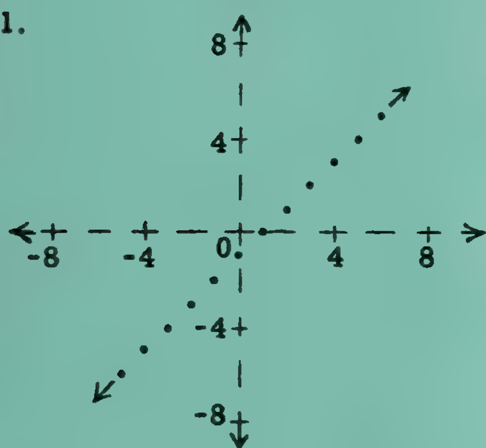
8. 5 [The graphs of (−3, −1), (−2, 0), (−1, 1), (0, 2), (1, 3).]

9. 4 [The graphs of (−3, 0), (−2, 1), (−1, 2), (0, 3).]

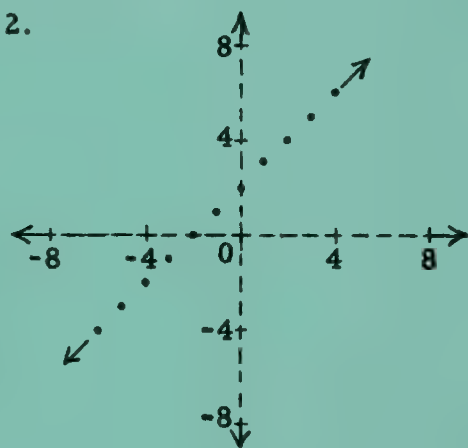
10. 3 [The graphs of (−1, −5), (0, −2), (1, 1).]

11. 0 [The students have found, for Exercise 10, which dots satisfy the first requirement of Exercise 11; the second requirement would be satisfied by the graphs of (−2, −4), (−1, −2), (0, 0), and (1, 2). But none of these would satisfy the first requirement.]

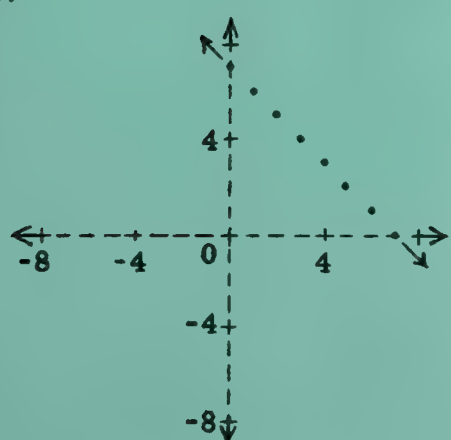
B. 1.



2.



3.



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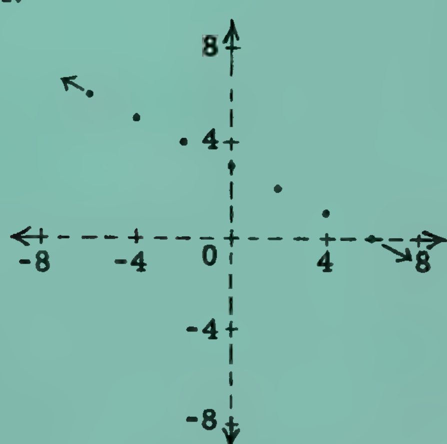
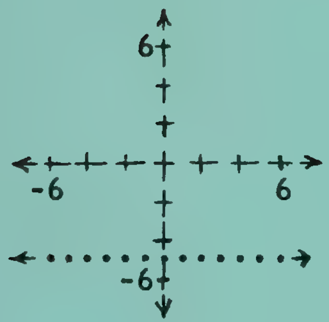


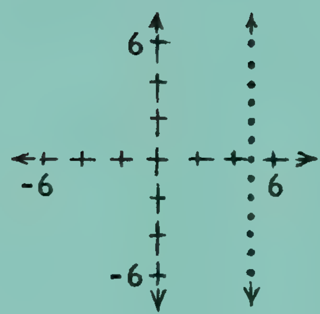
A.

4/170

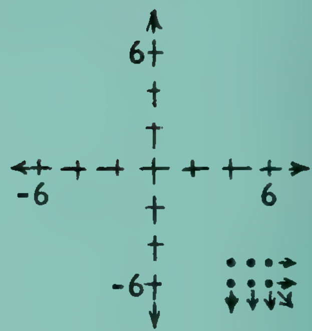
5.



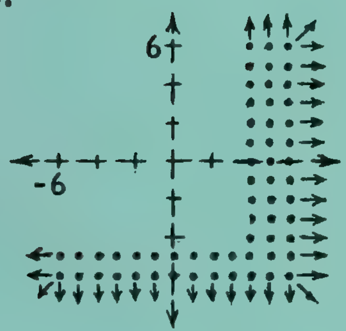
6.



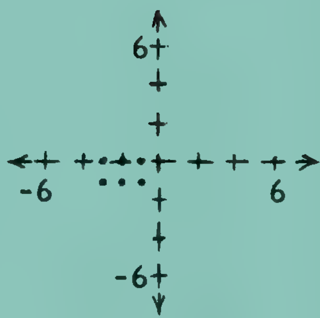
7.



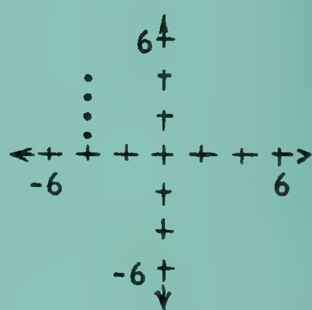
8.



9.



10.



C. 1. The set of all ordered pairs of integers with first component greater than −6 but less than −2 and second component greater than 1 but less than 5.

2. The set of all ordered pairs of integers such that the first component is equal to the opposite of the second component.

3. The set of all ordered pairs of integers such that the second component is equal to the opposite of three times the first component.

4. The set of all ordered pairs of integers such that the first component is two greater than the second.

5. second component is −5.

6. their graphs have first coordinate 5.

7. the first component is more than 3 <u>and</u> the second component is less than −4.

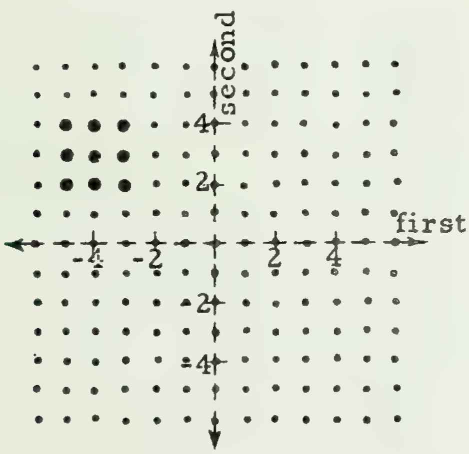
8. Repeat Exercise 7 using 'or' instead of 'and'.

9. the second component is less than 1 but greater than −2 <u>and</u> the first component is greater than −4 but less than 0. [How many points are there in this set?]

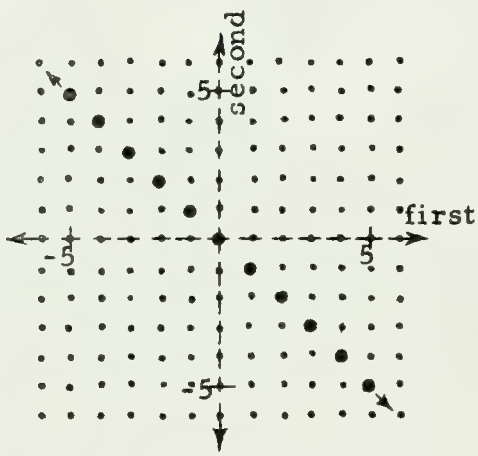
10. the second component is greater than 0 but less than 5 and the first component is −4.

<u>C</u>. Here are pictures of sets of ordered pairs of integers. Write descriptions of the sets pictured, using the type of wording of the exercises of Part B on page 4-6.

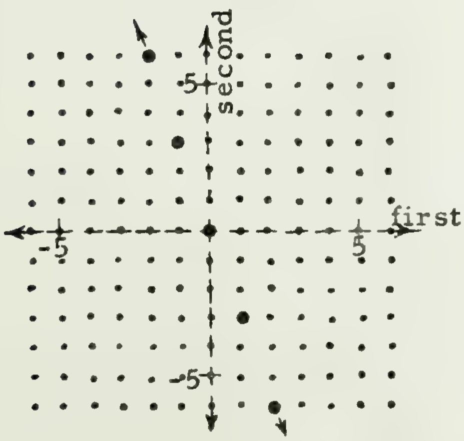
1.



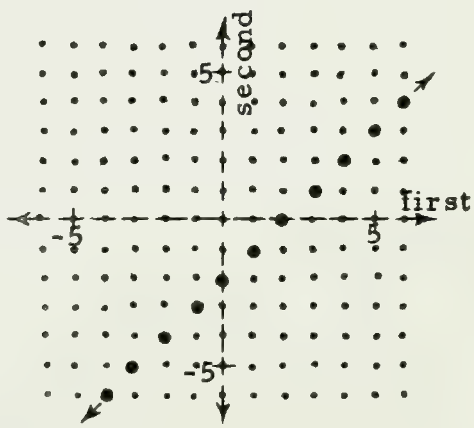
2.



3.



4.



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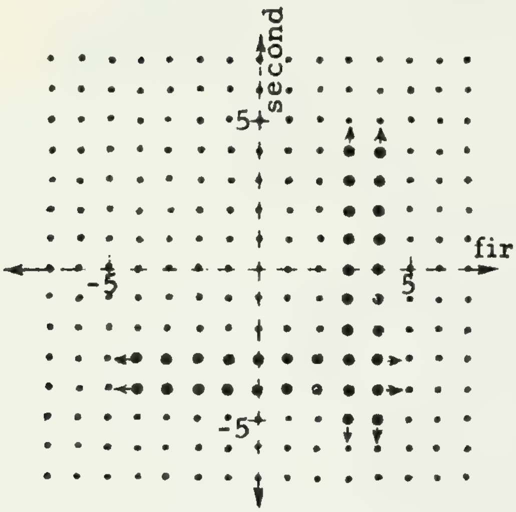
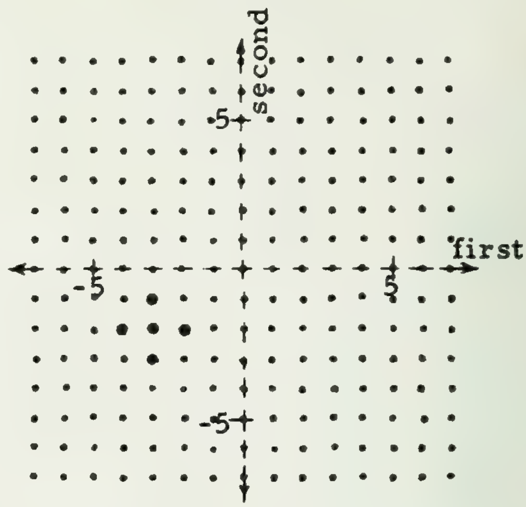
5.



6.

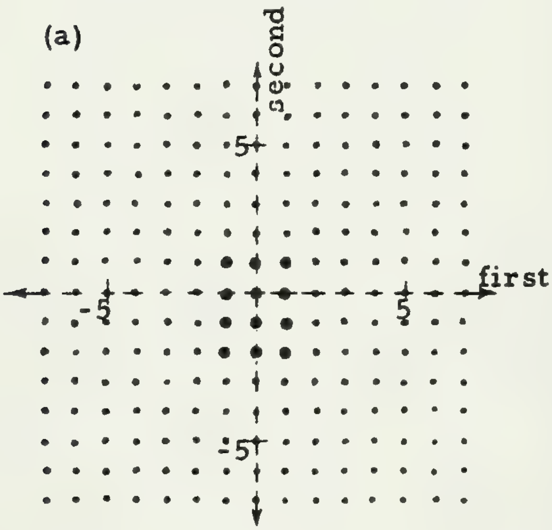


D. 1. Rewrite the descriptions of Part B of the Supplementary Exercises using the brace-notation mentioned in Part C of page 4-9.

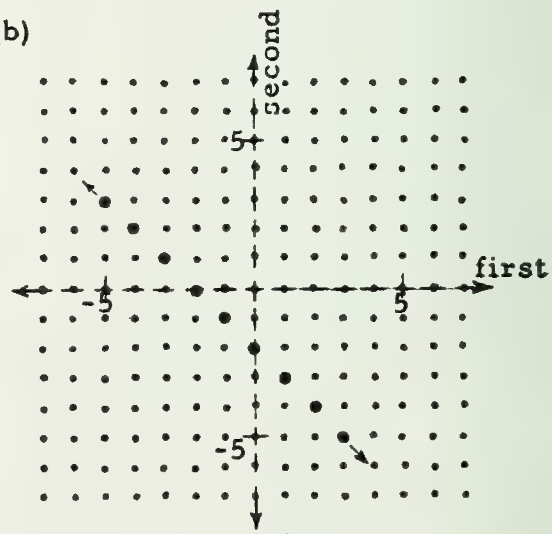
2. Rewrite the descriptions of the sets pictured in Part C of the Supplementary Exercises, using the brace-notation.

3. Describe the sets pictured below [by using the brace-notation].

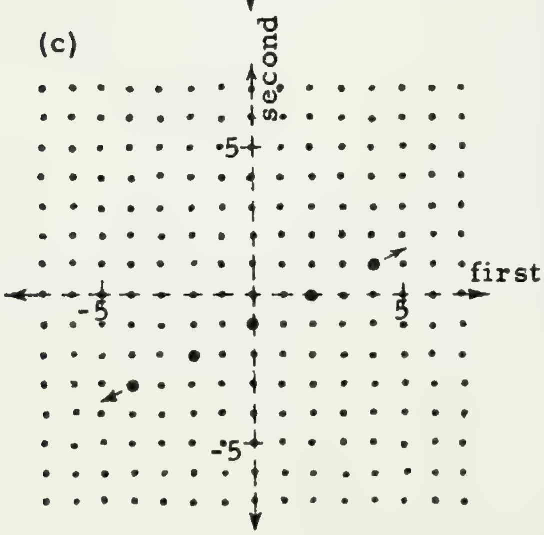
(a)



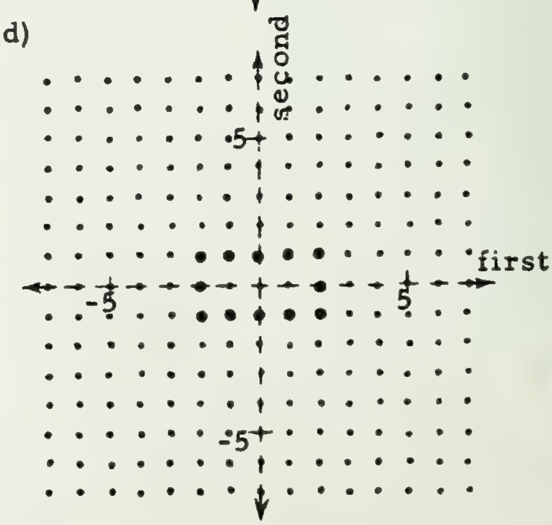
(b)



(c)



(d)



tion of a

5. The set of all ordered pairs of integers with first component greater than 2 but less than 5 or second component greater than −5 but less than −2.

6. The set of all ordered pairs of integers such that (the first component is −3 and the second component is greater than −4 and less than 0) or (the second component is −2 and the first component is greater than −5 and less than −1).

D. 1. (1) $\{(x, y),\ x \text{ and } y \text{ integers}:\ x = y + 1\}$

> For each of the other exercises in this part, <u>in order to save space</u>, we display the sentence to be written in place of '$x = y + 1$' in the answer for Exercise 1(1). It is not advisable to allow students this latitude at this time.

(2) $x = y - 2$ (3) $x + y = 7$ (4) $6 = x + 2y$

(5) $y = -5$ (6) $x = 5$ (7) $x > 3$ and $y < -4$

(8) $x > 3$ or $y < -4$ (9) $-2 < y < 1$ and $-4 < x < 0$

(10) $0 < y < 5$ and $x = -4$

2. (1) $-6 < x < -2$ and $1 < y < 5$ (2) $x = -y$ (3) $y = -3x$

(4) $x = y + 2$ (5) $2 < x < 5$ or $-5 < y < -2$

(6) $|x + 3| + |y + 2| \leq 1$, [or: ($x = -3$ and $-4 < y < 0$) or ($y = -2$ and $-5 < x < -1$)], [or: ($x = -2$ and $y = -2$) or ($x = -3$ and $-4 < y < 0$) or ($x = -4$ and $y = -2$)]

3. (a) $-2 < x < 2$ and $-3 < y < 2$ (b) $x + y = -2$

(c) $x = 2y + 2$ (d) ($|x| < 3$ and $|y| = 1$) or ($|x| = 2$ and $y = 0$)

5. . . .

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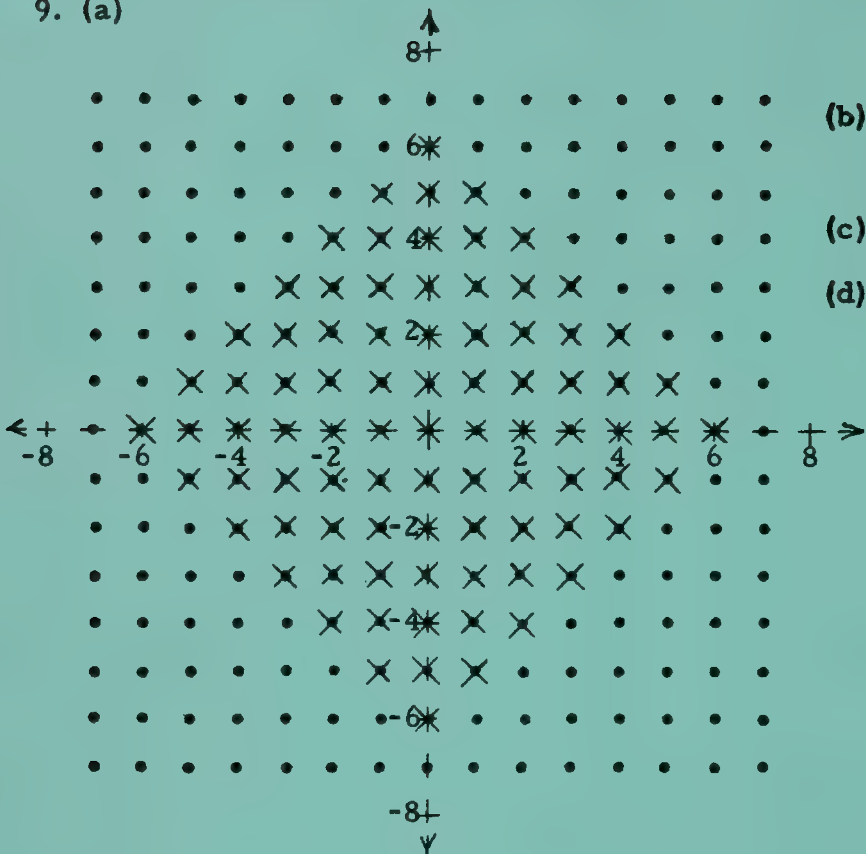
5. : :

9. (a)



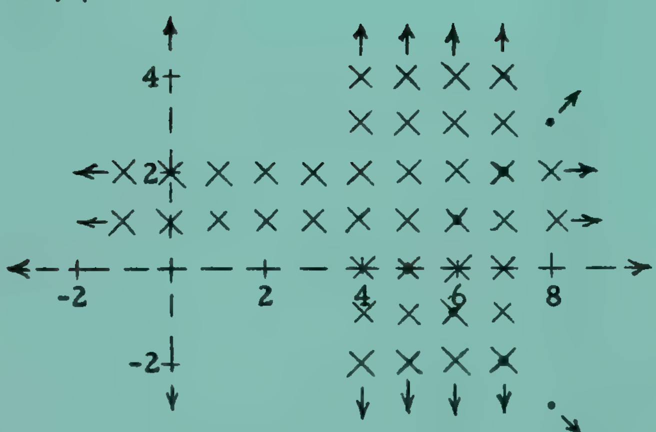
(b) n(R) = 225,

n(S) = 85

(c) $n(R \cap S) = 85$

(d) $n(R \cup S) = 225$

10. (a)



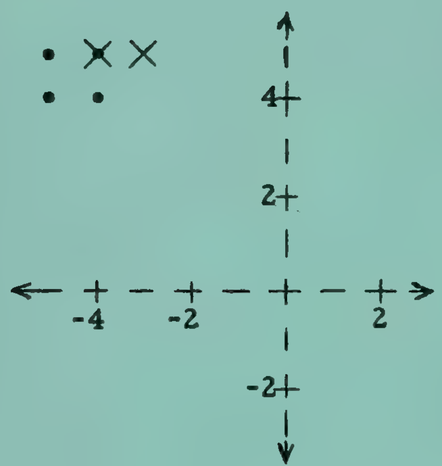
(b) Set R and Set S are infinite sets.

(c) $n(R \cap S) = 5$

(d) $n(R \cup S) =$ ____
This question does not make sense because the set is an infinite set.

tion of a

7. (a)

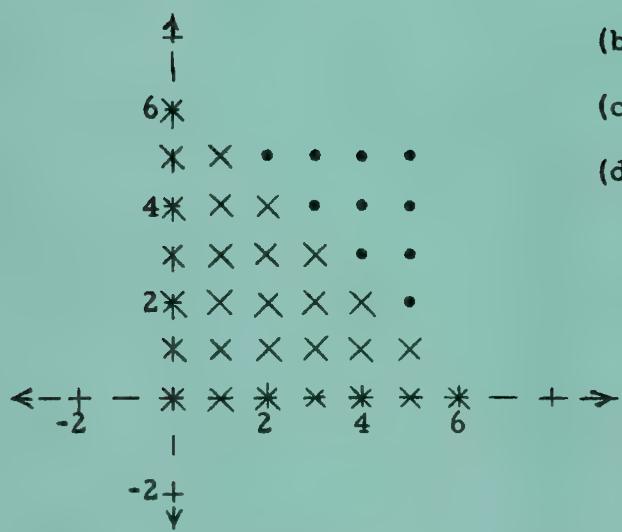


(b) $n(R) = 4$, $n(S) = 2$

(c) $n(R \cap S) = 1$

(d) $n(R \cup S) = 5$

8. (a)



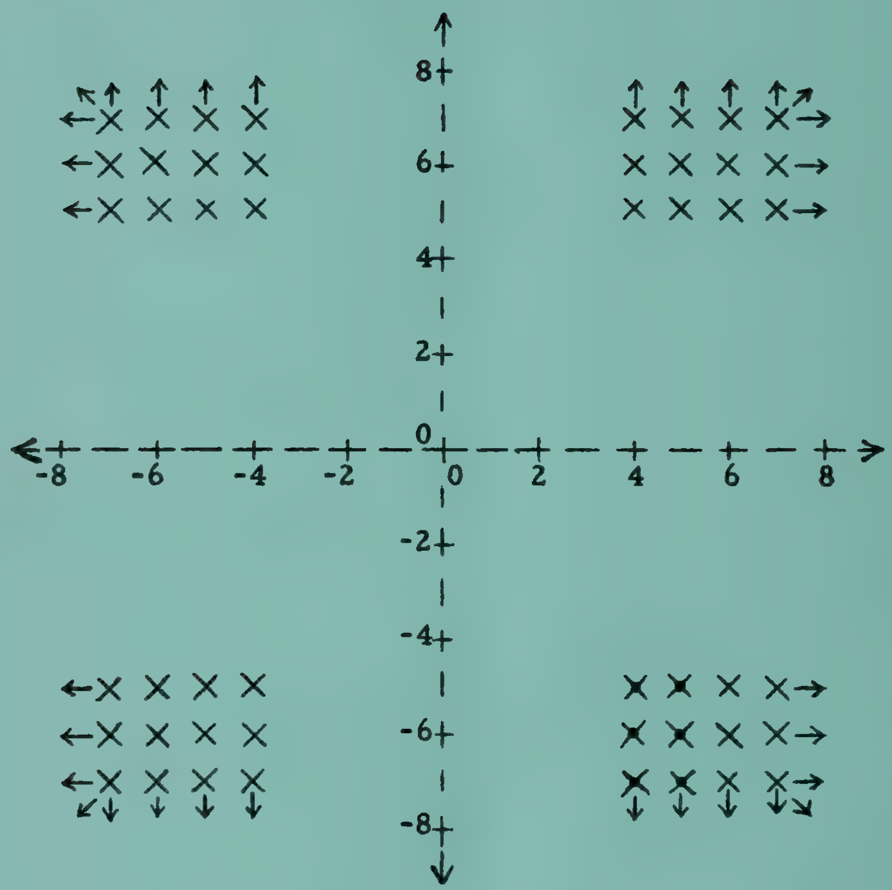
(b) $n(R) = 10$, $n(S) = 28$

(c) $n(R \cap S) = 0$

(d) $n(R \cup S) = 38$

ution of a

6. (a)



(b) n(R) = 6

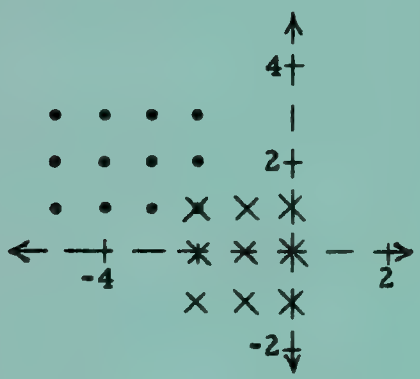
n(S) = ___ This question does not make sense because set S is an infinite set.

(c) $n(R \cap S) = 6$

(d) $n(R \cup S)$ = ___ This question does not make sense because the set is an infinite set.

ˑtion of a

3. (a)

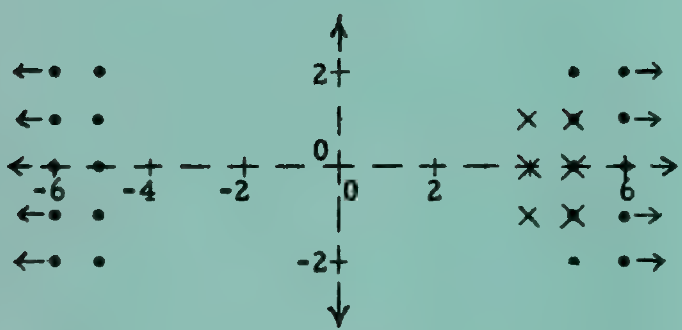


(b) n(R) = 12, n(S) = 9

(c) $n(R \cap S) = 1$

(d) $n(R \cup S) = 20$

4. (a)



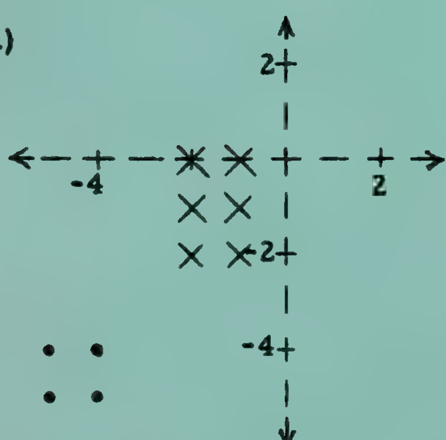
(b) n(R) = ____ This question does not make sense because set R is an infinite set.

n(S) = 6

(c) $n(R \cap S) = 3$

(d) $n(R \cup S) =$ ____ This question does not make sense because the set is an infinite set.

5. (a)



(b) n(R) = 4, n(S) = 6

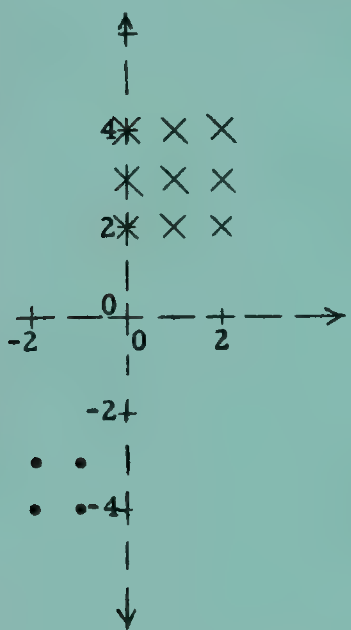
(c) $n(R \cap S) = 0$

(d) $n(R \cup S) = 10$

·tion of a

E. [In the sketches which follow, we have used ' • 's to indicate the points of set R and ' × 's to indicate the points of set S. The points in the intersection of sets R and S will therefore be indicated by ' ⋇ 's.]

1. (a)

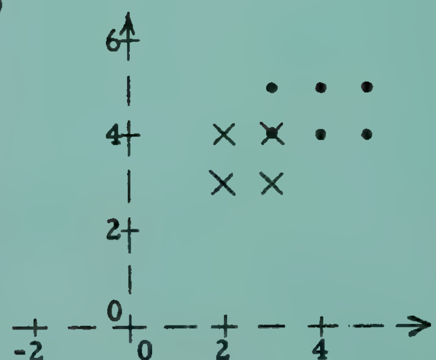


(b) $n(R) = 4,\ n(S) = 9$

(c) $n(R \cap S) = 0$

(d) $n(R \cup S) = 13$

2. (a)



(b) $n(R) = 6,\ n(S) = 4$

(c) $n(R \cap S) = 1$

(d) $n(R \cup S) = 9$

E. In each of the following exercises you are given a description of a set R and a set S. For each exercise,

(a) plot the points in each set on the same diagram,
(b) tell the number of points in each set,
(c) tell the number of points in the intersection, and
(d) tell the number of points in the union.

1. $R = \{(x, y),\ x \text{ and } y \text{ integers}:\ -3 < x < 0 \text{ and } -5 < y < -2\}$

   $S = \{(x, y),\ x \text{ and } y \text{ integers}:\ -1 < x < 3 \text{ and } 1 < y < 5\}$

2. $R = \{(x, y),\ x \text{ and } y \text{ integers}:\ 2 < x < 6 \text{ and } 3 < y < 6\}$

   $S = \{(x, y),\ x \text{ and } y \text{ integers}:\ 1 < x < 4 \text{ and } 2 < y < 5\}$

3. $R = \{(x, y),\ x \text{ and } y \text{ integers}:\ -6 < x < -1 \text{ and } 0 < y < 4\}$

   $S = \{(x, y),\ x \text{ and } y \text{ integers}:\ -3 < x < 1 \text{ and } -2 < y < 2\}$

4. $R = \{(x, y),\ x \text{ and } y \text{ integers}:\ |x| > 4 \text{ and } |y| < 3\}$

   $S = \{(x, y),\ x \text{ and } y \text{ integers}:\ 3 < x < 6 \text{ and } |y| < 2\}$

5. $R = \{(x, y),\ x \text{ and } y \text{ integers}:\ -6 < x < -3 \text{ and } -6 < y < -3\}$

   $S = \{(x, y),\ x \text{ and } y \text{ integers}:\ -3 < x < 0 \text{ and } -3 < y < 1\}$

6. $R = \{(4, -5), (4, -6), (4, -7), (5, -5), (5, -6), (5, -7)\}$

   $S = \{(x, y),\ x \text{ and } y \text{ integers}:\ |x| > 3 \text{ and } |y| > 4\}$

7. $R = \{(x, y),\ x \text{ and } y \text{ integers}:\ -6 < x < -3 \text{ and } 3 < y < 6\}$

   $S = \{(x, y),\ x \text{ and } y \text{ integers}:\ -5 < x < -2 \text{ and } 4 < y < 6\}$

8. $R = \{(x, y),\ x \text{ and } y \text{ integers}:\ x + y > 6,\ x < 6, \text{ and } y < 6\}$

   $S = \{(x, y),\ x \text{ and } y \text{ integers}:\ x + y \leq 6,\ x \geq 0, \text{ and } y \geq 0\}$

9. $R = \{(x, y),\ x \text{ and } y \text{ integers}:\ |x| \leq 7 \text{ and } |y| \leq 7\}$

   $S = \{(x, y),\ x \text{ and } y \text{ integers}:\ |x| + |y| \leq 6\}$

10. $R = \{(x, y),\ x \text{ and } y \text{ integers}:\ x - |y| = 5\}$

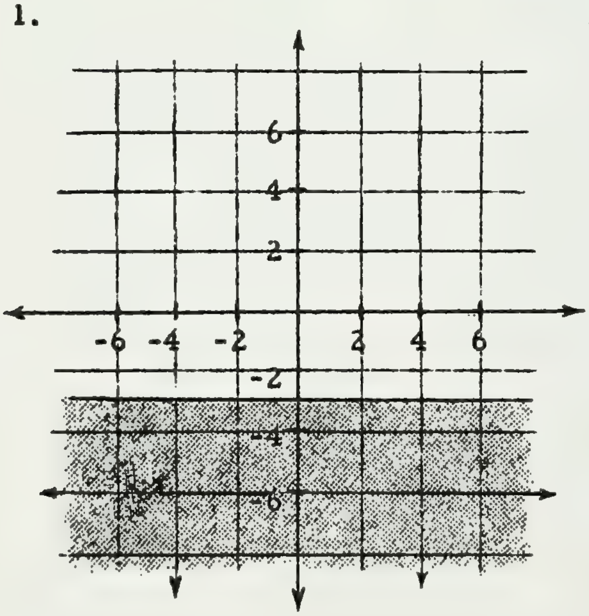
    $S = \{(x, y),\ x \text{ and } y \text{ integers}:\ 3 < x < 8 \text{ or } 1 \leq y < 3\}$

F. For each of the sets of ordered pairs described below, plot as many of the ordered pairs as you can on a picture of the number plane.

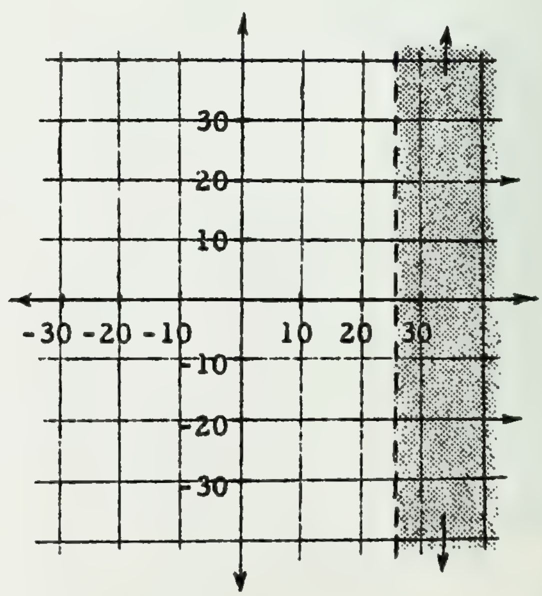
1. The set of all ordered pairs of real numbers such that the first component is equal to the sum of 2 and the product of -1 by the second component.
2. The set of all ordered pairs of real numbers such that the first component is 3 less than the second component.
3. $\{(x, y): \; 3x = 6 - y\}$
4. $\{(a, b): \; b = -a + 7\}$
5. The set of all ordered pairs of real numbers such that the second component is -2.
6. $\{(r, s): \; r > 1 \text{ and } s > -2\}$
7. $\{(s, r): \; s > 1 \text{ and } r > -2\}$
8. $\{(x, y): \; x \leq -1\}$
9. $\{(x, y): \; y \geq 4\}$
10. $\{(x, y): \; x \geq 3 \text{ or } y \leq 4\}$

G. For each set pictured below, write its description.

1.



2.



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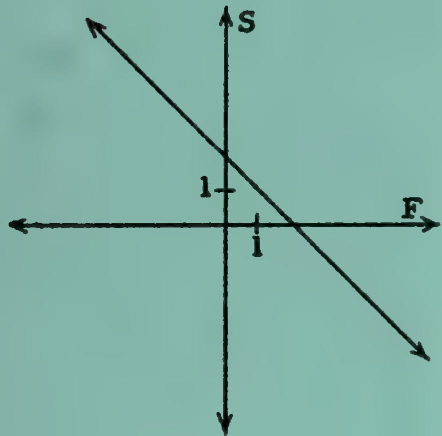

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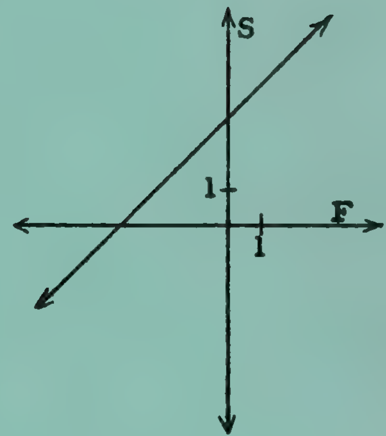

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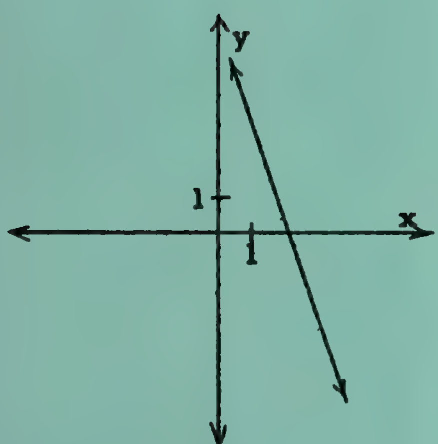

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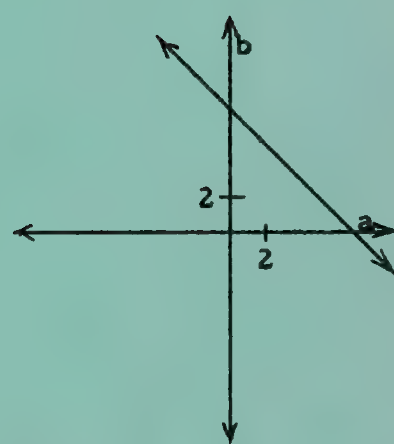

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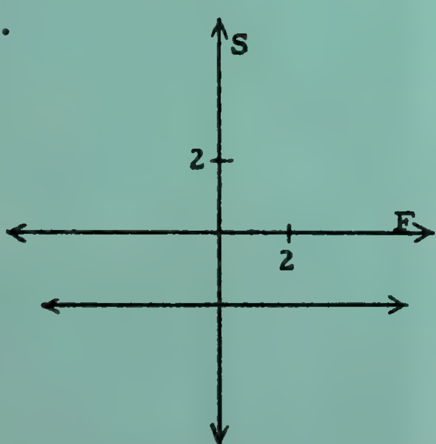

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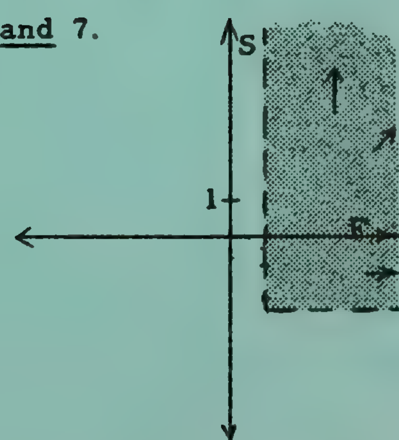

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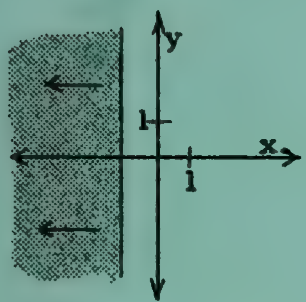

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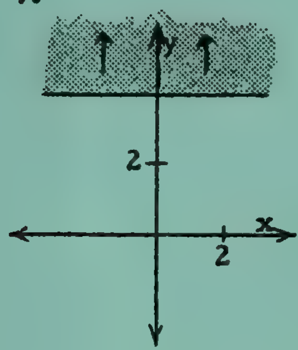

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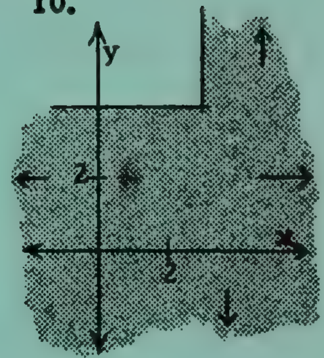


G. 1. $\{(x, y): y \leq -3\}$ 2. $\{(x, y): x > 25\}$

3. $\{(x, y): y > -9\}$ 4. $\{(x, y): x > 5 \text{ and } y < -5\}$

5. $\{(x, y): x \geq \frac{1}{2} \text{ or } y \geq \frac{3}{2}\}$ 6. $\{(x, y): y = -3x\}$

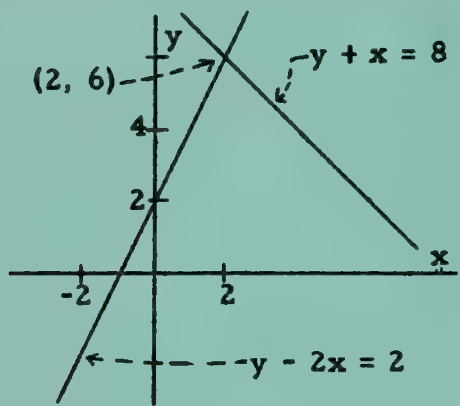
F. Fo

[4-123]

F. F

H. [See note regarding answers at the bottom of TC[4-35]a.]
[We assume all sets are infinite.]

1.



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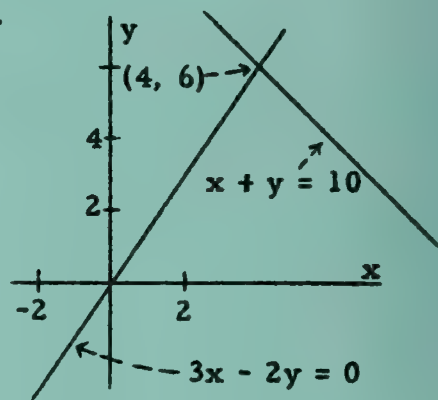

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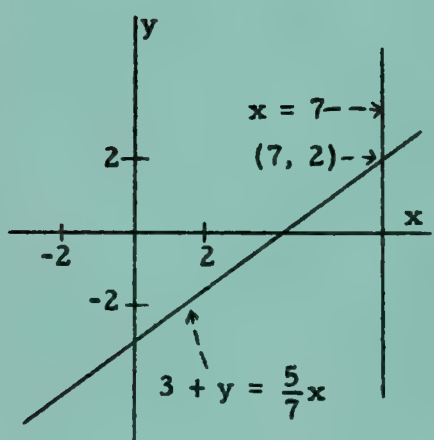

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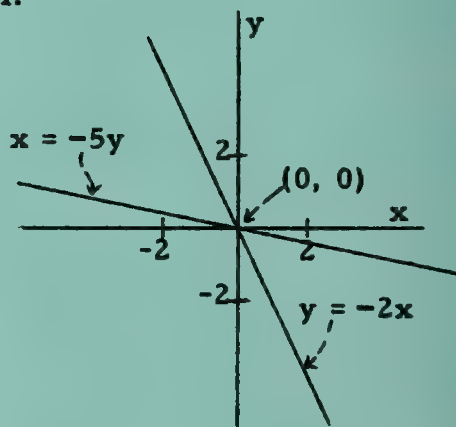
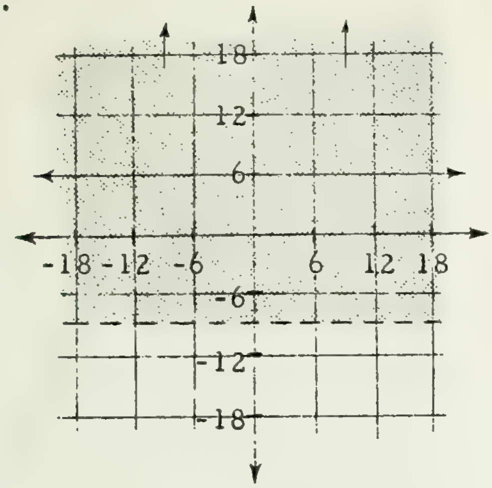


*

1. $\{(x, y): \ y - 2x = 2\} \cap \{(x, y): \ y + x = 8\} = \{(2, 6)\}$.
2. $\{(x, y): \ 3x - 2y = 0\} \cap \{(x, y): \ x + y = 10\} = \{(4, 6)\}$
3. $\{(x, y): \ x = 7\} \cap \{(x, y): \ 3 + y = \frac{5}{7}x\} = \{(7, 2)\}$
4. $\{(x, y): \ y = -2x\} \cap \{(x, y): \ x = -5y\} = \{(0, 0)\}$

*

3.



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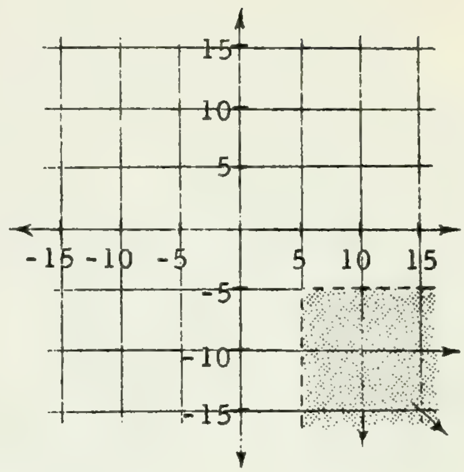

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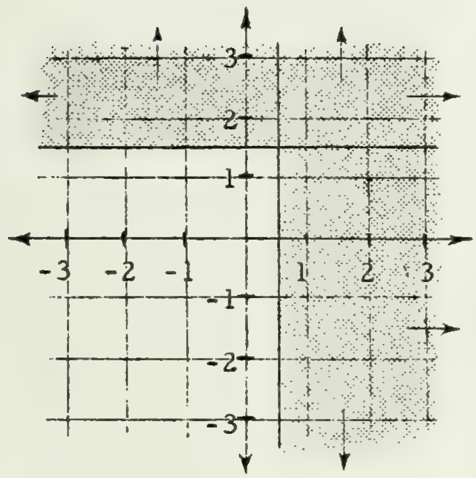

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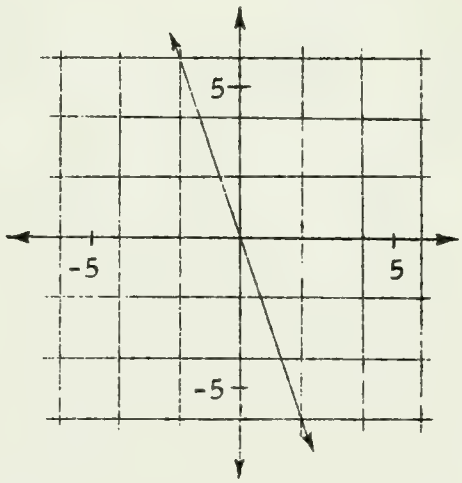


H. Each of the following exercises contains a pair of equations. Graph each of the two equations and tell the components of the points in the intersection of their solution sets.

1. (a) $y - 2x = 2$

   (b) $y + x = 8$

2. (a) $3x - 2y = 0$

   (b) $x + y = 10$

3. (a) $x = 7$

   (b) $3 + y = \frac{5}{7}x$

4. (a) $y = -2x$

   (b) $x = -5y$

(continued on next page)

5. (a) $3x - y = 7$
   (b) $4x + 3y = -8$

6. (a) $y + 2 = 3x$
   (b) $y = 3x + 4$

7. (a) $xx = 49$
   (b) $yy = 16$

8. (a) $|y| = 4$
   (b) $x - 3y = 3$

9. (a) $2x = 5 - y$
   (b) $\frac{1}{2}y = x + 7$

10. (a) $|x| = 3$
    (b) $|y| = 6\frac{1}{2}$

I. Complete each of these sentences to true ones in at least one way.

1. 4 is a factor of 56 with respect to the set of __________ because 4, 56, and ______ belong to this set and $56 = 4 \cdot$ ______.

2. -3 is a factor of 42 with respect to the set of __________ because -3, 42, and ______ belong to this set and $42 = -3 \cdot$ ______.

3. -8 is a factor of -72 with respect to the set of __________ because -8, -72, and ______ belong to this set and $-72 = -8 \cdot$ ______.

4. -9 is a factor of -12 with respect to the set of __________ because -9, -12, and ______ belong to this set and $-12 = -9 \cdot$ ______.

5. $\frac{1}{3}$ is a factor of 15 with respect to the set of __________ because $\frac{1}{3}$, 15, and ______ belong to this set and $15 = \frac{1}{3} \cdot$ ______.

6. $\frac{2}{3}$ is a factor of $\frac{3}{16}$ with respect to the set of __________ because $\frac{2}{3}$, $\frac{3}{16}$, and ______ belong to this set and $\frac{3}{16} = \frac{2}{3} \cdot$ ______.

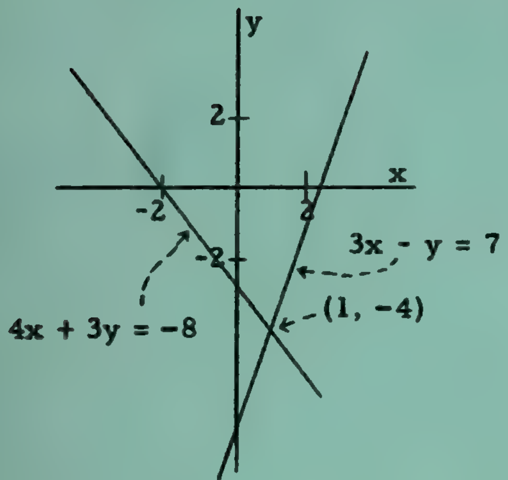
7. 9 is a factor of 81 with respect to the set of __________ because 9, 81, and ______ belong to this set and $81 = 9 \cdot$ ______.

8. $\sqrt{21}$ is a factor of 21 with respect to the set of __________ because $\sqrt{21}$, 21, and ______ belong to this set and $21 = \sqrt{21} \cdot$ ______.

because

se

5.



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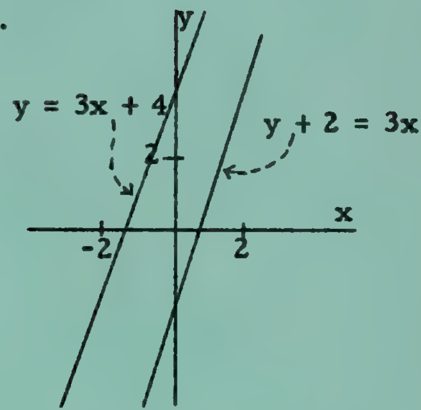
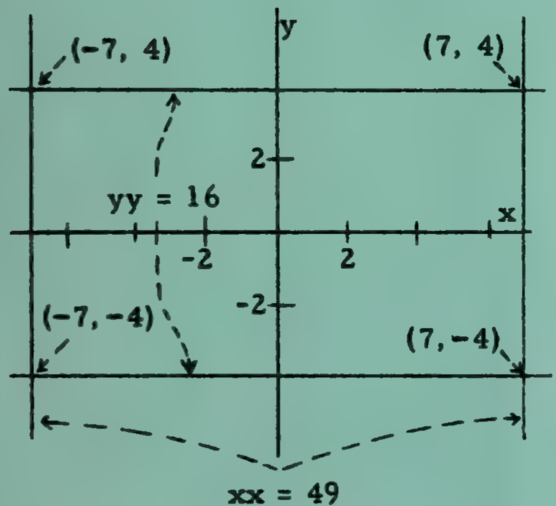


The intersection is $\emptyset$.

7.



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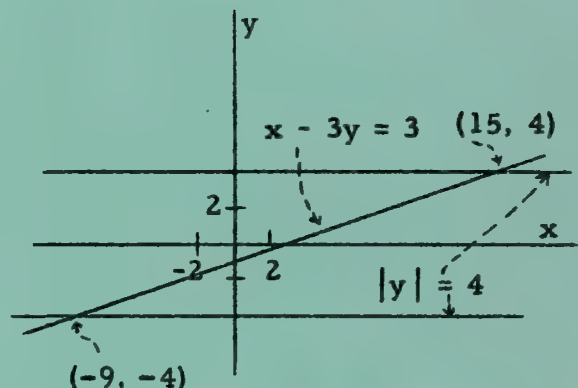


*

5. $\{(x, y): 3x - y = 7\} \cap \{(x, y): 4x + 3y = -8\} = \{(1, -4)\}$

6. $\{(x, y): y + 2 = 3x\} \cap \{(x, y): y = 3x + 4\} = \emptyset$

7. $\{(x, y): xx = 49\} \cap \{(x, y): yy = 16\} = \{(7, 4), (-7, 4), (-7, -4), (7, -4)\}$

8. $\{(x, y): |y| = 4\} \cap \{(x, y): x - 3y = 3\} = \{(-9, -4), (15, 4)\}$

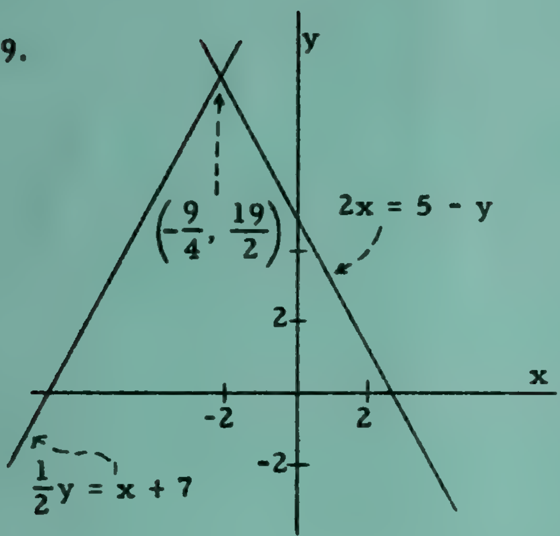
*

5.

because

se

9.



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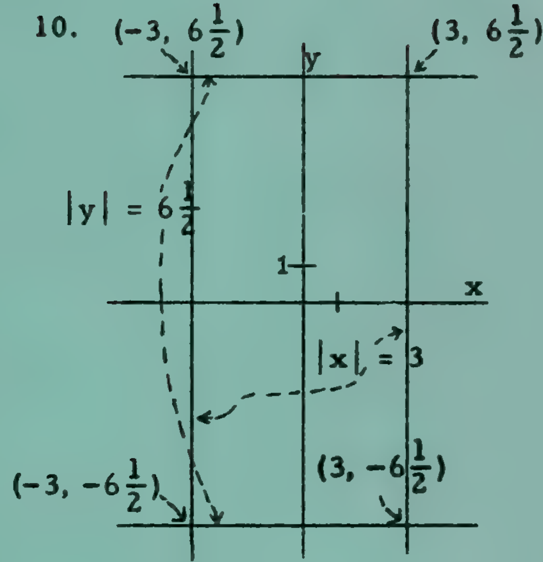

*

9. $\{(x, y)\colon\ 2x = 5 - y\} \cap \{(x, y)\colon\ \frac{1}{2}y = x + 7\} = \{(-\frac{9}{4}, \frac{19}{2})\}$

10. $\{(x, y)\colon\ |x| = 3\} \cap \{(x, y)\colon\ |y| = 6\frac{1}{2}\}$

$= \{(3, 6\frac{1}{2}), (-3, 6\frac{1}{2}), (-3, -6\frac{1}{2}), (3, -6\frac{1}{2})\}$

I. [See note on TC[4-49, 50].]

1. positive integers [or: reals, or: rationals], 14, 14
2. integers [or: reals, or: rationals], -14, -14
3. integers [or: reals, or: rationals], 9, 9
4. rationals [or: reals], $\frac{4}{3}$, $\frac{4}{3}$
5. rationals [or: reals], 45, 45
6. rationals [or: reals], $\frac{9}{32}$, $\frac{9}{32}$
7. positive integers [or: reals, or: rationals], 9, 9
8. reals, $\sqrt{21}$, $\sqrt{21}$

5.

[4-125]

because

se

5.

9. integers [or: reals], 0, 0

10. integers [or: reals, or: rationals], -1, -1

11. positive integers [or: reals, or: rationals], 578, 578

12. positive integers [or: reals, or: rationals], $78 + 22$, $78 + 22$

13. positive integers [or: reals, or: rationals], $4 \cdot 6$, $4 \cdot 6$

14. positive integers [or: reals, or: rationals], 7, 7, 7

J. 1. -3^3b^3 2. c^3d^3 3. $27^2 + r^4$

4. $a^5 - (-x)^4$ 5. r^4s^4 [or: $(rs)^4$]

6. $d^4g^2 - k^6$ [or $(d^2g)^2 - (k^3)^2$] 7. x^3y^4 [or: $(xy^2)^2x$]

8. $(r^2 + s)^2$ 9. $\dfrac{(-4)^3a^2 - b^4}{2^2a^3 + 8^4b}$

9. 52 is a factor of 0 with respect to the set of __________ because 52, 0, and _____ belong to this set and $0 = 52 \cdot$ ______.

10. 71 is a factor of −71 with respect to the set of __________ because 71, −71, and _______ belong to this set and $-71 = 71 \cdot$ _______.

11. 937 is a factor of $937 \cdot 578$ with respect to the set of __________ because 937, $937 \cdot 578$, and __________ belong to this set and $937 \cdot 578 = 937 \cdot$ __________.

12. 31 is a factor of $31 \cdot 78 + 31 \cdot 22$ with respect to the set of __________ because 31, $31 \cdot 78 + 31 \cdot 22$, and __________ belong to this set and $31 \cdot 78 + 31 \cdot 22 = 31 \cdot$ __________.

13. 17^2 is a factor of $(17 \cdot 4) \cdot (17 \cdot 6)$ with respect to the set of ____________ because 17^2, $(17 \cdot 4) \cdot (17 \cdot 6)$, and _____________ belong to this set and $(17 \cdot 4) \cdot (17 \cdot 6) = 17^2 \cdot$ __________.

14. _____^{2} is a factor of $21 \cdot 35$ with respect to the set of ____________ because ____^{2}, $21 \cdot 35$, and 15 belong to this set and $21 \cdot 35 =$ ___$^2 \cdot 15$.

J. Use exponents to abbreviate each expression.

1. $-3 \cdot -3 \cdot -3(bbb)$
2. $ccc(ddd)$
3. $27 \cdot 27 + rrrr$
4. $aaaaa - (-x \cdot -x \cdot -x \cdot -x)$
5. $(rs)(rs)(rs)(rs)$
6. $(ddg)(ddg) - (kkk)(kkk)$
7. $(xyy)(xyy)x$
8. $(rr + s)(rr + s)$
9. $\dfrac{-4 \cdot -4 \cdot -4(aa) - bbbb}{2 \cdot 2(aaa) + 8 \cdot 8 \cdot 8 \cdot 8b}$

K. Simplify.

1. $b^7 \cdot b^3$

2. $(-2)^4(-2)^7$

3. $4^2 \cdot 4 \cdot 4^4 \cdot 4^2$

4. $x^3 \cdot x^7 \cdot x^2 \cdot x^4$

5. $(2k^3)(5k^4)$

6. $(-2r^5)(5r^6)$

7. $(5xy^2)(-3x^2y)$

8. $(4ts^3)(-5t^3s^5)$

9. $(.3n^2r)(.7rn^3)$

10. $(\frac{1}{5}c^3d^5)(\frac{1}{7}c^5d^4)$

11. $\frac{k^7}{k^3}$

12. $\frac{5^5}{5^3}$

13. $\frac{-6^3}{6^2}$

14. $\frac{k^{10}}{-k^7}$

15. $\frac{9a^3b}{18a^5b^7}$

16. $\frac{70ts^3}{35ts^2}$

17. $\frac{45r^3t^5u^7}{-5u^6t^5r^7}$

18. $\frac{-100c^3de^5}{-5c^3d^2e^4}$

19. $\frac{18x^{70}y^{50}z^2}{-18x^{70}y^{50}z^2}$

20. $\frac{-5a^8b^8c}{10a^7b^7c^7}$

21. $(x^2)(x^5) - x^3$

22. $(x^3)(x^3) - x^3$

23. $(-5a^2)(-3a^3b^2)$

24. $(-7y^7)(-2yz^2)$

25. $(m^3n)(2mn^2)$

26. $(x_1^2x_2^3)(x_2^5x_1^3)$

27. $\frac{305^{98}}{305^{97}}$

28. $\frac{(-32)^5}{(-32)^6}$

29. $\frac{-105nrtu^5}{-1050n^2r^3t^4u^2}$

30. $\frac{-50x^2y^3z}{-25x^2y^2z^4}$

31. $\frac{24a^2bc^3}{-8x^2yz^3} \times \frac{15xy^2z}{-3ab^2c^2}$

32. $\frac{40mpr^3}{15k^2jt^3} \times \frac{-3kj^3t}{-4m^2pr^7}$

33. $\frac{2(r+t)^2(k+j)^3}{5xy} \times \frac{20x^3y^4}{4(r+t)^2}$

$^{-5})^4$

<u>K.</u> 1. b^{10} 2. -2^{11} [or: -2048] 3. 4^9 [or: 262144]

4. x^{16} 5. $10k^7$ 6. $-10r^{11}$

7. $-15x^3y^3$ 8. $-20t^4s^8$ 9. $0.21n^5r^2$

10. $\frac{1}{35}c^8d^9$ 11. k^4, $[k \neq 0]$ 12. 5^2 [or: 25]

13. -6 14. $-k^3$, $[k \neq 0]$ 15. $\frac{1}{2a^2b^6}$, $[ab \neq 0]$

16. $2s$, $[ts \neq 0]$ 17. $-\frac{9u}{r^4}$, $[utr \neq 0]$ 18. $\frac{20e}{d}$, $[cde \neq 0]$

19. -1, $[xyz \neq 0]$ 20. $\frac{ab}{-2c^6}$, $[abc \neq 0]$ 21. $x^7 - x^3$

22. $x^6 - x^3$ 23. $15a^5b^2$ 24. $14y^8z^2$

25. $2m^4n^3$ 26. $x_1^5x_2^8$ 27. 305

28. $-\frac{1}{32}$ 29. $\frac{u^3}{10nr^2t^3}$, $[nrtu \neq 0]$ 30. $\frac{2y}{z^3}$, $[xyz \neq 0]$

31. $\frac{15acy}{xz^2b}$, $[xyzabc \neq 0]$ 32. $\frac{2j^2}{kt^2mr^4}$, $[kjtmpr \neq 0]$

33. $2(k + j)^3x^2y^3$, $[xy(r + t) \neq 0]$ 34. x^{10}

35. $-y^6$ 36. z^{20} 37. 5^4t^8 [or: $625t^8$]

38. $-y^{171}$ 39. $x^{204}y^{204}$ [or: $(xy)^{204}$]

40. $2^5x^{10}y^{15}$ [or: $32x^{10}y^{15}$] 41. $3^2a^4b^8$ [or: $9a^4b^8$]

42. $3(5^2)a^5b^8$ [or: $75a^5b^8$] 43. $-4^5x^{21}y^{13}$ [or: $-1024x^{21}y^{13}$]

44. $-(\frac{1}{4})^3a^3$ [or: $-\frac{1}{64}a^3$] 45. $(\frac{1}{4})^2a^2$ [or: $\frac{1}{16}a^2$]

46. $\frac{x^5}{5}$ 47. $-\frac{t^5}{3^5}$ [or: $-\frac{t^5}{243}$] 48. x^{12}

49. $-0.5a^5$ 50. $2^3a^3b^9c^{12}$ [or: $8a^3b^9c^{12}$]

51. $3^2a^{14}b^2c^6$ [or: $9a^{14}b^2c^6$] 52. $-r^6t^9u^3$

53. $7^2t^2k^4s^2$ [or: $49t^2k^4s^2$] 54. $-y^{18}$

55. $-a^{40}$ 56. $128a^7b^{21}$ 57. $-x^{43}y^{18}z^{32}$

K. Simpl

[4-127]

-5)4

K. Sim

L. 1. nonequivalent [Using '2' for 'y', the corresponding value of '$(y^4)^3$' is $(2^4)^3$, that is 4096. The corresponding value of 'y^7' is 2^7, or 128.]

2. nonequivalent [Using '1' for 'a', the corresponding value of '$(3a)^7$' is 3^7, or 2187. The corresponding value of '$3a^7$' is 3.]

3. equivalent

4. nonequivalent [Using '2' for 'x' and '1' for 'y', the corresponding value of 'x^4y^7' is $2^4 \cdot 1^7$, or 16. The corresponding value of '$(xy)^{11}$' is $(2 \cdot 1)^{11}$, or 2048.]

5. equivalent

6. nonequivalent [Using '1' for 'x', the corresponding value of '$(-x)^4$' is $(-1)^4$, that is, 1. The corresponding value of '$-x^4$' is -1^4, or -1.]

7. nonequivalent [Using '1' for 'x' and '2' for 'y', the corresponding value of '$(x^2 + y^3)^2$' is $(1^2 + 2^3)^2$, or 81. The corresponding value of '$x^4 + y^6$' is $1^4 + 2^6$, or 65.]

8. equivalent

9. nonequivalent [Using '1' for 'a', the corresponding value of '$\frac{3 + a^2}{a^2}$' is $\frac{3 + 1^2}{1^2}$, that is, 4. The corresponding value of '3' is 3.]

10. equivalent with respect to $\{a: a \neq 0\}$

11. equivalent with respect to $\{y: y \neq 0\}$

34. $(x^2)^5$
35. $(-y^2)^3$
36. $(-z^5)^4$
37. $(-5t^2)^4$
38. $(-y)^{171}$
39. $(-xy)^{204}$
40. $(2x^2y^3)^5$
41. $(-3a^2b^4)^2$
42. $(3a^3b^4)(5ab^2)^2$
43. $(-xy^3)(4x^4y^2)^5$
44. $(-\frac{1}{4}a)^3$
45. $(-\frac{1}{4}a)^2$
46. $(\frac{x}{5})^3(5x)^2$
47. $(-\frac{t}{3})^2(-\frac{t}{3})^3$
48. $(\frac{1}{8}x^5)^2(-8x)^2$
49. $(-0.5a)^3(-2a)^2$
50. $(2ab^3c^4)^3$
51. $(-3a^7bc^3)^2$
52. $(-r^2t^3u)^3$
53. $(7tk^2s)^2$
54. $(-y^2)^5(y^4)^2$
55. $(-a^5)^4(-a^4)^5$
56. $(4a^2b^3)^2(2ab^5)^3$
57. $(-x^2yz)^4(-x^5y^2z^4)^7$

L. Each exercise contains two pronumeral expressions. For each pair of expressions, tell whether the expressions are equivalent or nonequivalent. If they are nonequivalent, give a counter-example.

1. $(y^4)^3$, y^7
2. $(3a)^7$, $3a^7$
3. $(ab)^3$, a^3b^3
4. x^4y^7, $(xy)^{11}$
5. $(-x)^3$, $-x^3$
6. $(-x)^4$, $-x^4$
7. $(x^2+y^3)^2$, x^4+y^6
8. $(x^2y^3)^2$, x^4y^6
9. $\frac{3+a^2}{a^2}$, 3
10. $\frac{3a^2}{a^2}$, 3
11. $\frac{27y^4 - 21y^2x^2}{15y^2}$, $\frac{9y^2 - 7x^2}{5}$

M. Simplify.

1. $4m^3 \cdot -5m^2$
2. $-3x^3 \cdot -6x^4$
3. $-5y^4 \cdot 7y^5$
4. $-z \cdot 12z^2$
5. $8mt^2 \cdot -6mt$
6. $-5z^3x \cdot 7x^2z^2$
7. $-4n^2p^3 \cdot 6n^4p^2$
8. $6t \cdot -\frac{1}{2}s$
9. $-\frac{2}{5}a \cdot -3a^2b$
10. $-12rs^2 \cdot -\frac{4}{3}s^2r$
11. $-4a^2b^3 \cdot -4a^3$
12. $-0.2x^2y \cdot -0.4xy^3$
13. $-42x^3 \div (7x^2)$
14. $-66y^5 \div (6y^2)$
15. $40a^4 \div (-2a)$
16. $-22m^2n \div (-11mn)$
17. $44a^2b^2 \div (-4b^2)$
18. $18m^3n^2 \div (6m^3n^2)$
19. $5t(4t + 7)$
20. $9k(2k - 5)$
21. $-5m(m^2 - 2m)$
22. $-7c^2(10c - 4c^2)$
23. $pq(p + q)$
24. $-xy(y - x)$
25. $3r(r^2 - 2r + 8)$
26. $8s(7 - 2s + 6s^2)$
27. $-3t(2t^2 - 7t + 9)$
28. $-12m(8 - 3m + 6m^2)$
29. $4xy(x^2 + xy + y^2)$
30. $-3ab(4a^2 - 6ab + 7b^2)$
31. $24\left(\frac{1}{3}p^2 - \frac{1}{4}p + \frac{7}{8}\right)$
32. $-56\left(\frac{1}{7}u^2 - \frac{3}{8}u + \frac{1}{14}\right)$
33. $4m^2(3m^2 - 7m + 2)$
34. $-9k^2(k^2 - 7k - 3)$
35. $-5cd(3d^2 - 4cd + 9c^2)$
36. $xyz(x^2 + y^2 + z^2)$
37. $m^2n^2p^2(3mn - 4mp + 3np)$
38. $5r^2s^2t^2(5rt + 3sr - 2st)$
39. $(30m + 40n) \div 5$
40. $(15k + 36n) \div 3$
41. $(48a - 24b) \div -4$
42. $(20m^3n^2 - 30m^2n^3) \div (-20m^2n^2)$
43. $(28k^2m - 84km^2) \div 28mk$
44. $(2x^3y^2 - xy) \div (-xy)$
45. $(10m^2 - 30m + 50) \div -10$
46. $(-6p^2 - 7a + 3) \div -1$
47. $(30p^4 - 15p^3 + 20p^2) \div (5p^2)$
48. $\dfrac{(p + 7)^2 + 8(p + 7)}{p + 7}$
49. $\dfrac{2(x + y + 1)^2 - 5(x + y + 1)}{-(x + y + 1)}$
50. $\dfrac{3(x + 1)(y + 2z) - 4(x + 1)^2(y + 2z)}{(x + 1)(y + 2z)}$

N. Factor.

1. $7a + 7b$
2. $6h - 6j$
3. $tm + tn$
4. $rx - yr$
5. $2bx + 2by$
6. $tz - t$

M.

1. $-20m^5$ 2. $18x^7$ 3. $-35y^9$ 4. $-12z^3$

5. $-48m^2t^3$ 6. $-35x^3z^5$ 7. $-24n^6p^5$ 8. $-3ts$

9. $\frac{6}{5}a^3b$ 10. $16r^2s^4$ 11. $16a^5b^3$ 12. $0.08x^3y^4$

13. $-6x$, $[x \neq 0]$ 14. $-11y^3$, $[y \neq 0]$ 15. $-20a^3$, $[a \neq 0]$

16. $2m$, $[mn \neq 0]$ 17. $-11a^2$, $[b \neq 0]$ 18. 3, $[mn \neq 0]$

19. $20t^2 + 35t$ 20. $18k^2 - 45k$ 21. $-5m^3 + 10m^2$

22. $-70c^3 + 28c^4$ 23. $p^2q + pq^2$ 24. $-xy^2 + x^2y$

25. $3r^3 - 6r^2 + 24r$ 26. $56s - 16s^2 + 48s^3$

27. $-6t^3 + 21t^2 - 27t$ 28. $-96m + 36m^2 - 72m^3$

29. $4x^3y + 4x^2y^2 + 4xy^3$ 30. $-12a^3b + 18a^2b^2 - 21ab^3$

31. $8p^2 - 6p + 21$ 32. $-8u^2 + 21u - 4$

33. $12m^4 - 28m^3 + 8m^2$ 34. $-9k^4 + 63k^3 + 27k^2$

35. $-15cd^3 + 20c^2d^2 - 45c^3d$ 36. $x^3yz + xy^3z + xyz^3$

37. $3m^3n^3p^2 - 4m^3n^2p^3 + 3m^2n^3p^3$

38. $25r^3s^2t^3 + 15r^3s^3t^2 - 10r^2s^3t^3$

39. $6m + 8n$ 40. $5k + 12n$

41. $-12a + 6b$ 42. $-m + \frac{3}{2}n$, $[mn \neq 0]$

43. $k - 3m$, $[km \neq 0]$ 44. $-2x^2y + 1$, $[xy \neq 0]$

45. $-m^2 + 3m - 5$ 46. $6p^2 + 7a - 3$

47. $6p^2 - 3p + 4$, $[p \neq 0]$ 48. $p + 15$, $[p \neq -7]$

49. $-2x - 2y + 3$, $[x + y \neq -1]$ 50. $-4x - 1$, $[x \neq 1, y \neq -2z]$

N.

1. $7(a + b)$ 2. $6(h - j)$ 3. $t(m + n)$

4. $r(x - y)$ 5. $2b(x + y)$ 6. $t(z - 1)$

M. Simplify

1.

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72y

M. Simplif 4/186

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7. $6(2r - t)$ 8. $10(3c + 2d)$ 9. $24(2x - 3y)$

10. $x(y^2 - 2)$ 11. $7p^2(p^2 + 1)$ 12. $(1/2)h(a + b)$

13. $\pi(r^2 + R^2)$ 14. $3(p^2 - 3)$ 15. $ab(b - 2a)$

16. $25uv(2 - 3uv)$ 17. $11m^2n(3mn - 2)$ 18. $37(59 + 41)$

19. $2(x^2 + 4x + 2)$ 20. $3(p^2 - 2p - 20)$ 21. $y(a + b - ab)$

22. $(z + 9)(z - 9)$ 23. $(4 + p)(4 - p)$ 24. $(3 + q)(3 - q)$

25. $4(36 + c^2)$ 26. $(5m + n)(5m - n)$ 27. $(n + 2s)(n - 2s)$

28. $(m^2 + 3)(m^2 - 3)$ 29. $[s + (t/4)][s - (t/4)]$

30. $[5 + (k/8)][5 - (k/8)]$ 31. $(j + 12)(j - 12)$

32. $(x^2 + 0.25)(x + 0.5)(x - 0.5)$ 33. $\pi(R + r)(R - r)$

34. $(5m + 8n)(5m - 8n)$ 35. $(7x_1 + 8x_2)(7x_1 - 8x_2)$

36. $(10r_1 + 9r_2)(10r_1 - 9r_2)$ 37. $(p + 3)(p + 1)$

38. $(u + 6)(u + 3)$ 39. $(m + 9)(m + 3)$

40. $3(c + 5)(c + 3)$ 41. $5(y + 7)(y + 5)$ 42. $2(8 + x)(3 + x)$

43. $3y(y + 12)(y + 3)$ 44. $2z(z^3 + 13z^2 + 40)$

45. $3(2m + 3n)(2m - 3n)$ 46. $70(3c + 1)(3c - 1)$

47. $a(x + 2)(x + 1)$ 48. $y^2(5x^4 - 16)$

49. $(x - 1)(x + 4)$, [or: $[(x + 3) - 4][(x + 3) + 1]$]

50. $(a + b - 7)(a + b + 1)$ 51. $(2x + y + 7)(2x + y - 3)$

52. $(k - p + 3)(k - p - 2)$ 53. $(2r + 2s + 1)(r + s + 2)$

54. $(3x - 3y - 2)(x - y + 2)$ 55. $(x+y)(2x+3y+z)(2x+y-z)$

56. $(3k - 2)(3k + 22)(3k + 20)$ 57. $4xy$

58. $(10a - 7b)(-4a + 3b)$

O. 1. $ab(2a^2 + 3b)$ 2. $4xy^3(4x - 3y)$ 3. $2p^2m^3(4 + m)$

4. $3abc^2(2a^2b + 5c^5)$ 5. $3rs(rs + 1)$

6. $xy(7z - 6 - 7yz)$ 7. $2x(y + 4z + 2x)$ 8. $5x^2(1 + 5x + 6x^7)$

7. $12r - 6t$
8. $30c + 20d$
9. $48x - 72y$
10. $xy^2 - 2x$
11. $7p^4 + 7p^2$
12. $\frac{1}{2}ha + \frac{1}{2}hb$
13. $\pi r^2 + \pi R^2$
14. $3p^2 - 9$
15. $ab^2 - 2a^2b$
16. $50uv - 75u^2v^2$
17. $33m^3n^2 - 22m^2n$
18. $59 \cdot 37 + 41 \cdot 37$
19. $2x^2 + 8x + 4$
20. $3p^2 - 6p - 60$
21. $ay + yb - aby$

22. $z^2 - 81$
23. $16 - p^2$
24. $9 - q^2$
25. $144 + 4c^2$
26. $25m^2 - n^2$
27. $n^2 - 4s^2$
28. $m^4 - 9$
29. $s^2 - (t^2/16)$
30. $25 - (k^2/64)$
31. $j^2 - 144$
32. $x^4 - 0.0625$
33. $\pi R^2 - \pi r^2$
34. $25m^2 - 64n^2$
35. $49x_1^2 - 64x_2^2$
36. $100r_1^2 - 81r_2^2$

37. $p^2 + 4p + 3$
38. $u^2 + 9u + 18$
39. $m^2 + 12m + 27$
40. $3c^2 + 24c + 45$
41. $5y^2 + 60y + 175$
42. $48 + 22x + 2x^2$
43. $3y^3 + 45y^2 + 108y$
44. $2z^4 + 26z^3 + 80z$
45. $12m^2 - 27n^2$
46. $630c^2 - 70$
47. $ax^2 + 3ax + 2a$
48. $5y^2x^4 - 16y^2$
49. $(x + 3)^2 - 3(x + 3) - 4$
50. $(a + b)^2 - 6(a + b) - 7$
51. $(2x + y)^2 + 4(2x + y) - 21$
52. $(k - p)^2 + (k - p) - 6$
53. $2(r + s)^2 + 5(r + s) + 2$
54. $3(x - y - 1)^2 + 10(x - y - 1) + 3$
55. $4(x + y)^3 - (x + y)(y + z)^2$
56. $9(3k - 2)(k + 7)^2 - (3k - 2)$
57. $(x + y)^2 - (x - y)^2$
58. $(3a - 2b)^2 - (7a - 5b)^2$

O. Use the idea of the HCF in factoring these expressions.

1. $2a^3b + 3ab^2$
2. $16x^2y^3 - 12xy^4$
3. $8m^3p^2 + 2p^2m^4$
4. $6a^3b^2c^2 + 15abc^7$
5. $3r^2s^2 + 3rs$
6. $7xyz - 6xy - 7xy^2z$
7. $2xy + 8xz + 4x^2$
8. $5x^2 + 25x^3 + 30x^9$

P. Find an LCM of the given pronumeral expressions.

1. $a,\ 3b,\ 5c$
2. $10m,\ 30mn,\ 30nr$
3. $k,\ k^4,\ k^8$
4. $3y^3,\ 6y,\ y^7$
5. $5km^3,\ 15k^3m$
6. $1,\ 3xy,\ 27x^2y$
7. $4cd^2,\ 12c^2d$
8. $9j,\ 18,\ 27jk$
9. $3x^2y^3z,\ 9x^3yz^2,\ -18xy^2z^3$
10. $6g^2h^2,\ 3c^2d^2,\ 12cdjh$
11. $r + s,\ 13$
12. $5(a + b),\ 15$
13. $r + s - 2t,\ 14$
14. $12k - 24,\ 36$
15. $6(x + y),\ 18x + 18y$
16. $2(3a - b),\ (b - 3a)^2$
17. $2r - 3s,\ 2r + 3s$
18. $7 + 5k,\ 5k + 7$
19. $3(a - 2b),\ 2(a - 3b)$
20. $x + y,\ x^2 - y^2$
21. $x^8 - y^8,\ x^4 + y^4,\ x - y$
22. $a + 2,\ a - 2,\ a^2 - 4$
23. $x + 1,\ 2x - 1$
24. $7x^2 + x - 1,\ x^2 + x - 7$
25. $(a + b)(a + 3b)^2(a - b),\ (a - b)^2(3a + b),\ (a + b)^2$

Q. Reduce these fractions.

1. $\frac{7p}{7q}$
2. $\frac{18c}{36d}$
3. $\frac{7r}{14r}$
4. $\frac{9p^2}{11p^2}$
5. $\frac{xy}{zy}$
6. $\frac{mt^3}{nt^2}$
7. $\frac{a^2b}{a^2c}$
8. $\frac{4xm^2}{8ym^2}$
9. $(12p^3)/(4p)$
10. $(27z^4)/(9z^2)$
11. $(8r^2)/(25r^3)$
12. $(7ab^2)/(21a^2b)$
13. $\frac{39t^4s^2}{52ts^3}$
14. $\frac{12m^2n}{-8mp}$
15. $\frac{-30c^2d^2}{-70cd^2}$
16. $\frac{-48k^3r^3}{48r^3x^3}$
17. $\frac{9(y + 4)}{11(y + 4)}$
18. $\frac{35(3 - m)}{55(3 - m)}$
19. $\frac{18(4 - x)}{27(x - 4)}$
20. $\frac{p^2(a + b)}{p(a + b)^2}$
21. $\frac{8y^2(r - 2s)^2}{4y(r - 2s)^2}$
22. $\frac{11(k - 9)}{11k}$
23. $\frac{3p}{3(p + 3)}$
24. $\frac{2y(m + n)}{16y^2}$
25. $\frac{7j(j + 2)}{7j}$
26. $\frac{xy + xz}{uy + uz}$

P. 1. $15abc$ 2. $30mnr$ 3. k^8 4. $6y^7$

5. $15k^3m^3$ [or: $15(km)^3$] 6. $27x^2y$

7. $12c^2d^2$ [or: $12(cd)^2$] 8. $54jk$

9. $18x^3y^3z^3$ [or: $18(xyz)^3$] 10. $12c^2d^2g^2h^2j$ [or: $12(cdgh)^2j$]

11. $13(r + s)$ 12. $15(a + b)$ 13. $14(r + s - 2t)$

14. $36(k - 2)$ 15. $18(x + y)$ 16. $2(3a - b)^2$

17. $(2r - 3s)(2r + 3s)$ 18. $5k + 7$

19. $6(a - 2b)(a - 3b)$ 20. $(x + y)(x - y)$

21. $(x^4 + y^4)(x^2 + y^2)(x - y)(x + y)$ [or: $x^8 - y^8$]

22. $(a + 2)(a - 2)$ [or: $a^2 - 4$] 23. $(x + 1)(2x - 1)$

24. $(7x^2 + x - 1)(x^2 + x - 7)$ 25. $(a + b)^2(a + 3b)^2(a - b)^2(3a + b)$

Q. [See note at bottom of page 4-84.]

1. $\frac{p}{q}$ 2. $\frac{c}{2d}$ 3. $\frac{1}{2}$ 4. $\frac{9}{11}$

5. $\frac{x}{z}$ 6. $\frac{mt}{n}$ 7. $\frac{b}{c}$ 8. $\frac{x}{2y}$

9. $3p^2$ 10. $3z^2$ 11. $\frac{8}{25r}$ 12. $\frac{b}{3a}$

13. $\frac{3t^3}{4s}$ 14. $\frac{3mn}{-2p}$ 15. $\frac{3c}{7}$ 16. $\frac{-k^3}{x^3}$

17. $\frac{9}{11}$ 18. $\frac{7}{11}$ 19. $-\frac{2}{3}$ 20. $\frac{p}{a + b}$

21. $2y$ 22. $\frac{k - 9}{k}$ 23. $\frac{p}{p + 3}$ 24. $\frac{m + n}{8y}$

25. $j + 2$ 26. $\frac{x}{u}$

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27. $\frac{s}{r}$	28. $\frac{5}{x + 1}$	29. $\frac{y - 7}{7}$	30. $\frac{8 - r}{3}$
31. $\frac{x + y}{x - y}$	32. $\frac{y + 2}{y - 2}$	33. $\frac{p - 1}{3}$	34. $x - 3$
35. $\frac{1}{y + 1}$	36. $\frac{k}{k - 1}$	37. $\frac{a}{1 - a}$	38. $20x$
39. $5k$	40. $7q$	41. $25x$	52. $\frac{8}{de}$
43. $\frac{n}{m}$	44. $\frac{2a^3}{b^4}$	45. $\frac{12m^2d}{e}$	46. $m^3a^2p^2$
47. $\frac{2}{3(y - 2)}$	48. $\frac{a}{4(a - 3)}$	49. $\frac{8x}{y(x + y)}$	50. $\frac{3m - 1}{6}$
51. $x(a + 3)$	52. $\frac{4k(x + y)}{5(x - y)}$	53. $\frac{acd^2}{4b}$	54. $\frac{n^4}{m^2}$
55. $\frac{ab}{8c}$	56. $\frac{4}{x - 1}$	57. $2(m + 6)$	58. $\frac{9p}{q(p + q)}$
59. $-\frac{1}{3}$	60. -1	61. $3y$	62. $\frac{5 - 2y}{y^2}$
63. $\frac{1 - 7p}{p^2}$	64. $\frac{2s - 3r^2}{r^2s}$	65. $\frac{2dx + 2cy}{c^2d^2}$	66. $\frac{3(x - 1)}{8}$
67. $\frac{4a^2 - 8a + 5}{10a^3}$	68. $\frac{-2(m + 6)}{m^2 - 9}$	69. $\frac{8x + 42}{x^2 - 25}$	70. $\frac{8t}{t^2 - 4}$

27. $\frac{sp - sq}{rp - rq}$ 28. $\frac{5x - 5}{x^2 - 1}$ 29. $\frac{y^2 - 49}{7y + 49}$

30. $\frac{64 - r^2}{24 + 3r}$ 31. $\frac{(x + y)^2}{x^2 - y^2}$ 32. $\frac{y^2 - 4}{(y - 2)^2}$

33. $\frac{6p^2 - 6}{18p + 18}$ 34. $\frac{x^2 - 3x}{x}$ 35. $\frac{y}{y^2 + y}$

36. $\frac{k^2 - 3k}{k^2 - 4k + 3}$ 37. $\frac{6a - a^2}{6 - 7a + a^2}$

*

Simplify.

38. $\frac{1}{2} \cdot 40x$ 39. $\frac{5}{k} \cdot k^2$ 40. $pq \cdot \frac{7}{p}$

41. $10x^2 \cdot \frac{5}{2x}$ 42. $ed \cdot \frac{8}{e^2d^2}$ 43. $\frac{12m}{7n} \cdot \frac{28n^2}{48m^2}$

44. $\frac{8a^2}{5b^2} \cdot \frac{10ab}{8b^3}$ 45. $\frac{36m^3d^2}{5e^3} \cdot \frac{15e^2}{9md}$ 46. $\frac{m^5}{a^2p^3} \cdot \frac{a^4p^5}{m^2}$

47. $\frac{5}{y^2 - 4} \cdot \frac{2y + 4}{15}$ 48. $\frac{4a + 12}{16a} \cdot \frac{a^2}{a^2 - 9}$ 49. $\frac{8x - 8y}{xy^2} \cdot \frac{x^2y}{x^2 - y^2}$

50. $\frac{9m^2 - 1}{12} \cdot \frac{2}{1 + 3m}$ 51. $\frac{(a + 3)^2}{x^5} \cdot \frac{5x^6}{5a + 15}$ 52. $\frac{12k^3}{(x - y)^2} \cdot \frac{x^2 - y^2}{15k^2}$

53. $\frac{5ab^2}{8cd} \div \frac{10b^3}{4c^2d^3}$ 54. $\frac{mn^2}{m^2n} \div \frac{m}{n^3}$ 55. $\frac{3a^2b^2}{8c} \div (3ab)$

56. $\frac{16}{x^2 - 1} \div \frac{4}{x + 1}$ 57. $\frac{m^2 - 36}{21} \div \frac{m - 6}{42}$ 58. $\frac{9p - 9q}{pq^2} \div \frac{p^2 - q^2}{p^2q}$

59. $\frac{4 - 2r}{r^2 - 7r + 10} \cdot \frac{r^2 - 25}{6r + 30}$ 60. $\frac{a - b}{a + b} \cdot \frac{b^2 - a^2}{b^2 - 2ab + a^2}$

61. $\frac{5xr^2 - 15xr - 50x}{xr^2 - 25x} \div \frac{10ry + 20y}{6y^2r + 30y^2}$

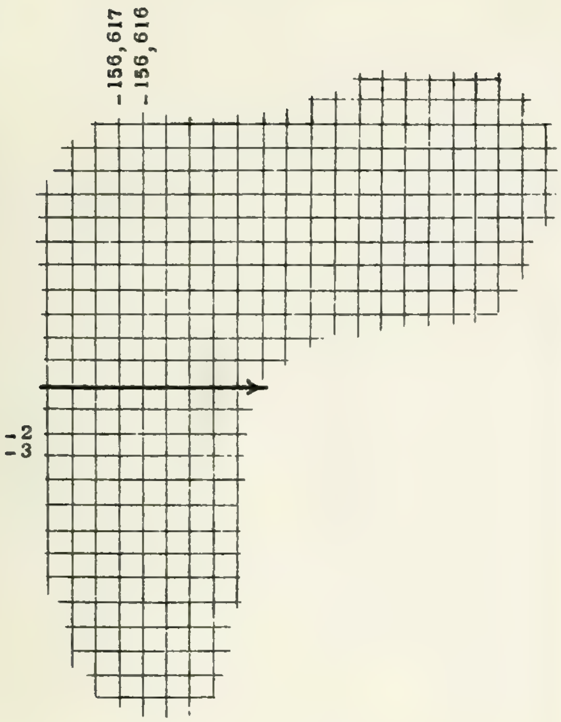
62. $\frac{5}{y^2} - \frac{2}{y}$ 63. $\frac{1}{p^2} - \frac{7}{p}$ 64. $\frac{2}{r^2} - \frac{3}{s}$

65. $\frac{2x}{c^2d} + \frac{2y}{cd^2}$ 66. $\frac{x - 2}{4} + \frac{x + 1}{8}$ 67. $\frac{2a + 1}{5a^2} - \frac{4a - 2}{4a^3}$

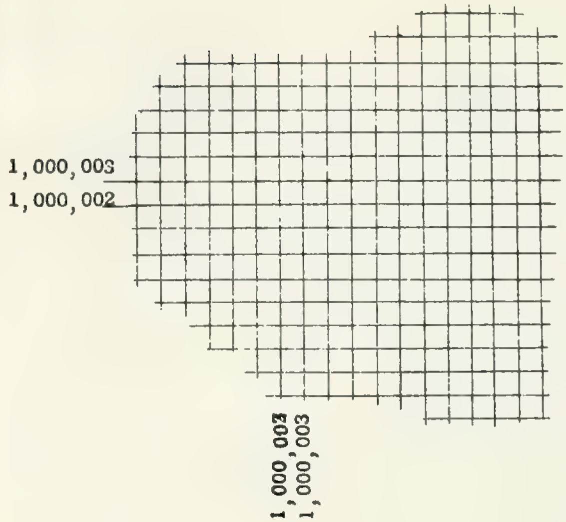
68. $\frac{2m}{m^2 - 9} - \frac{4}{m - 3}$ 69. $\frac{2}{x^2 - 25} - \frac{8}{5 - x}$ 70. $\frac{t + 2}{t - 2} - \frac{t - 2}{t + 2}$

NAME ______________________________

Region A



Region B



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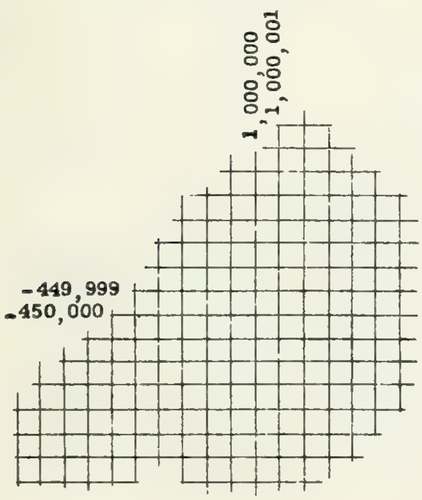

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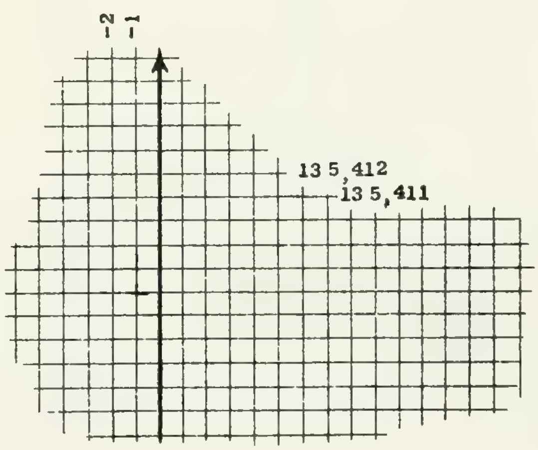

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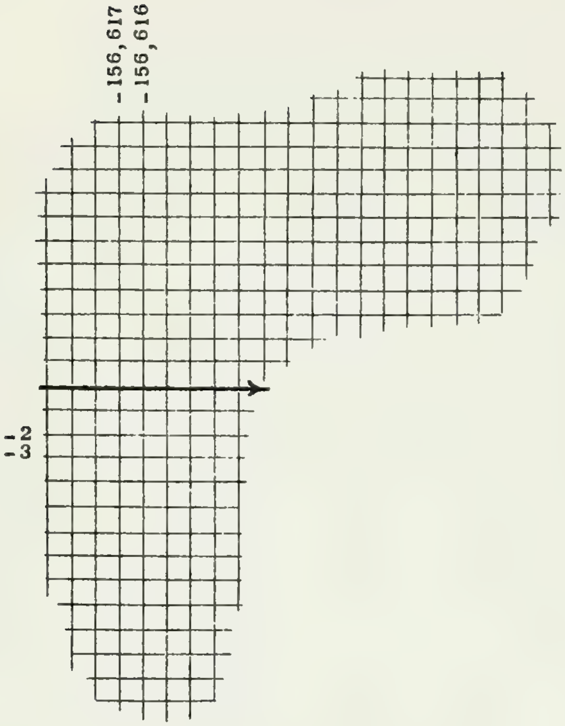

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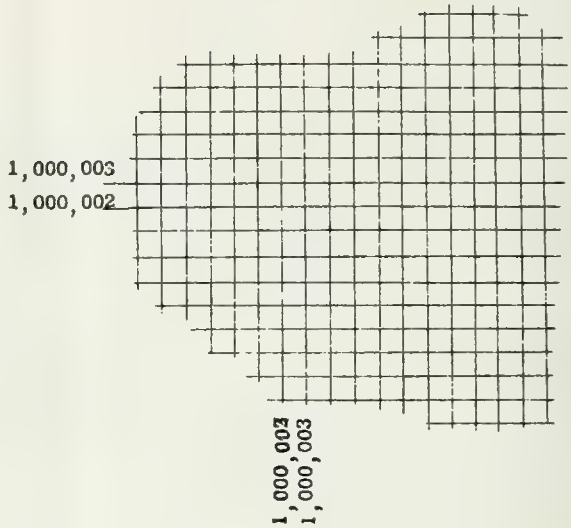

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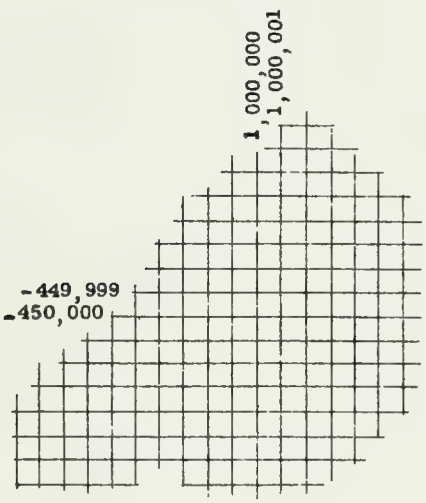

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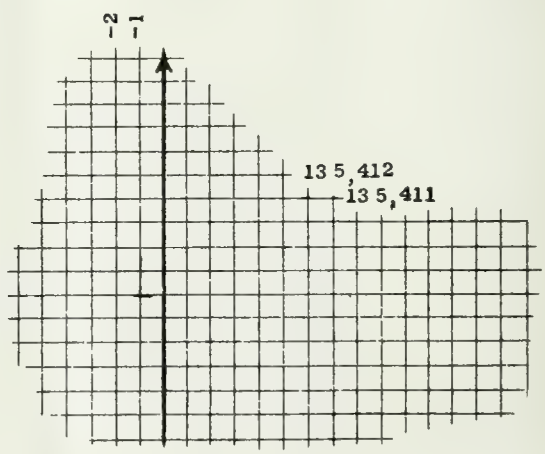
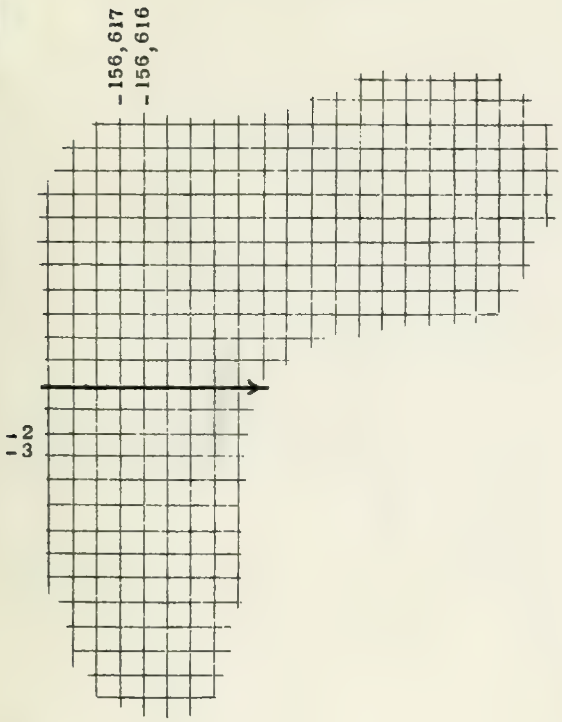


NAME ____________________

Region A



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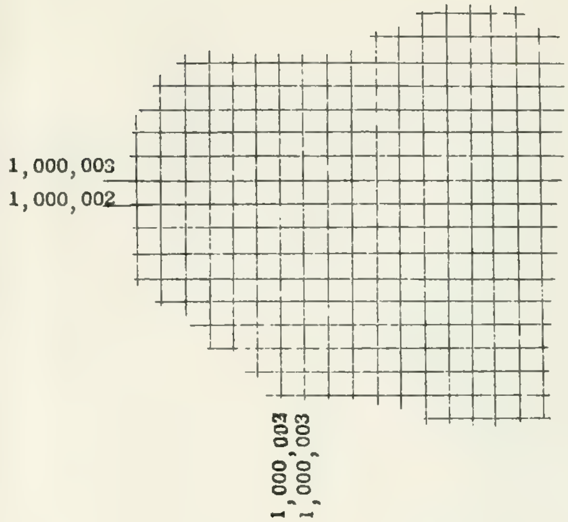
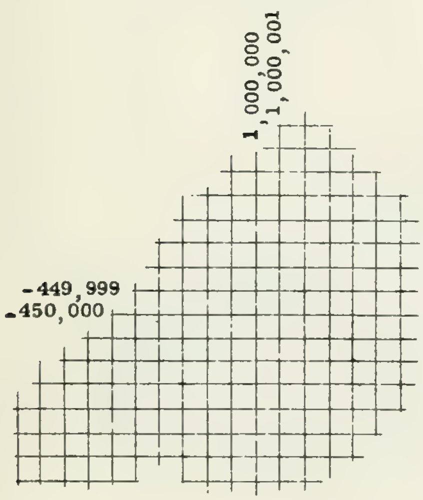
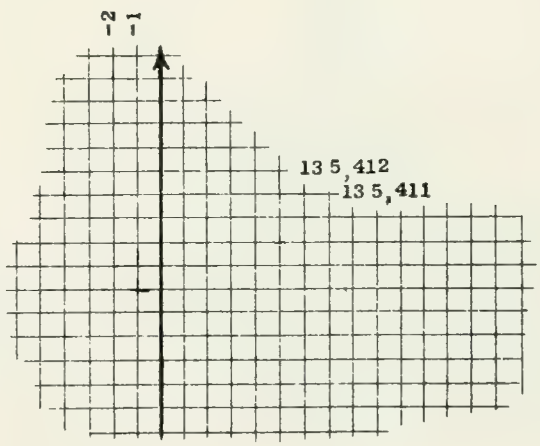


Region C



Region D



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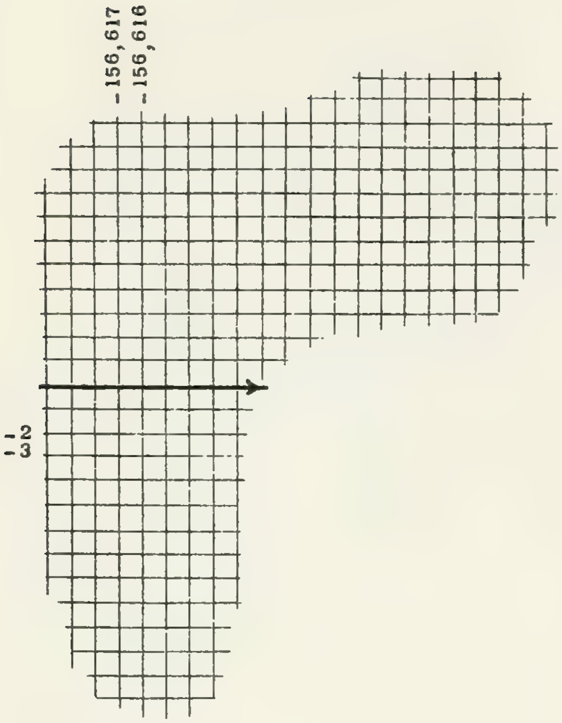

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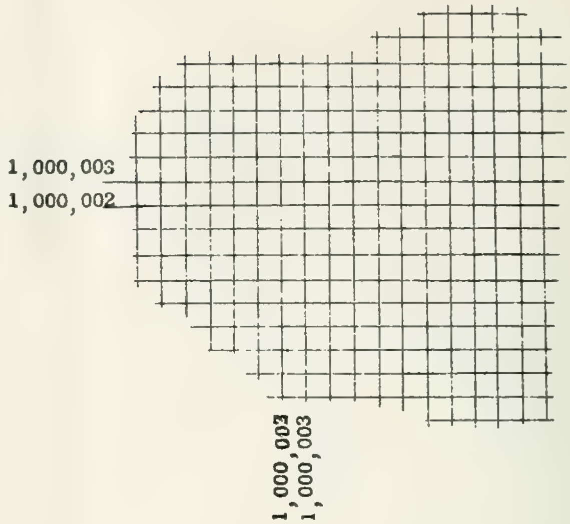

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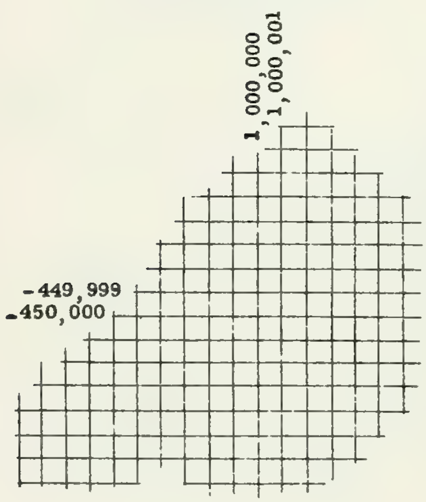

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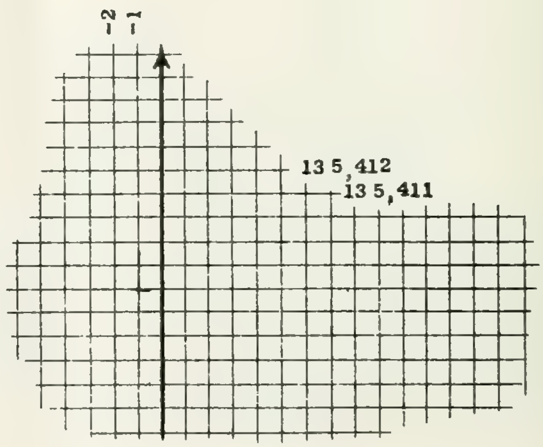

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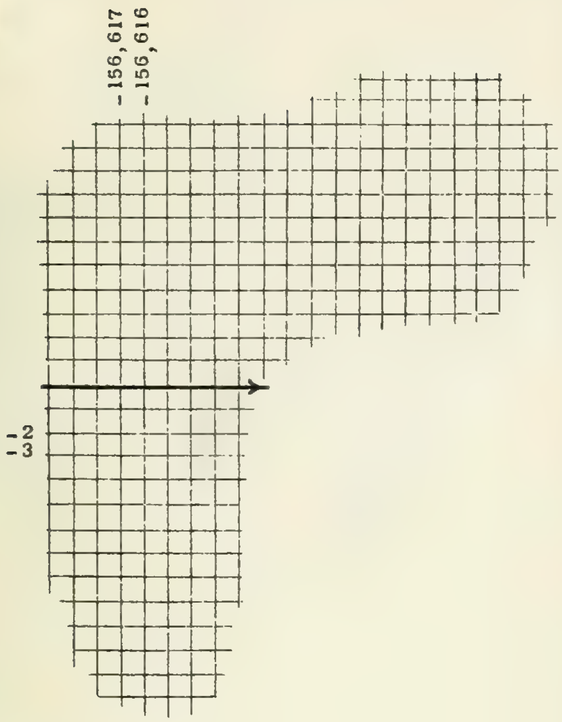

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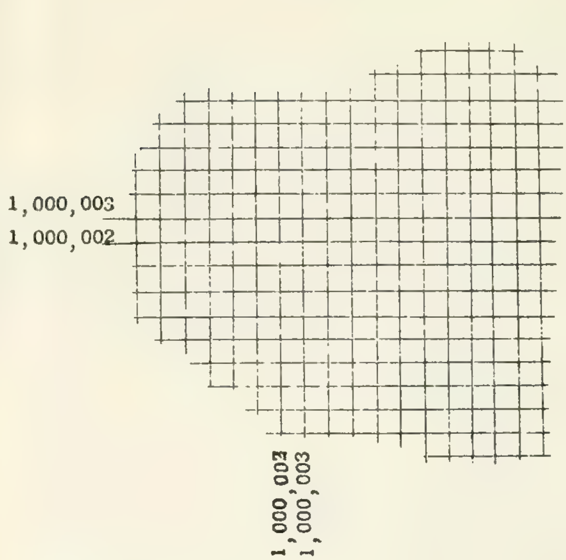

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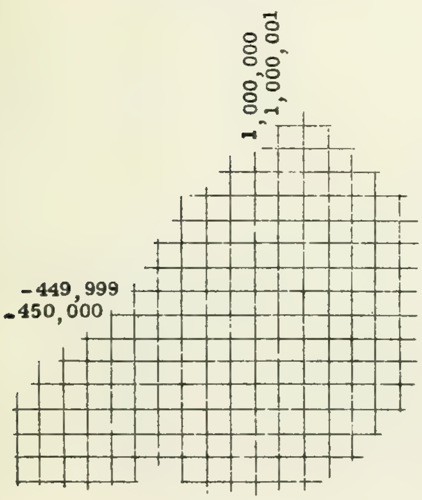

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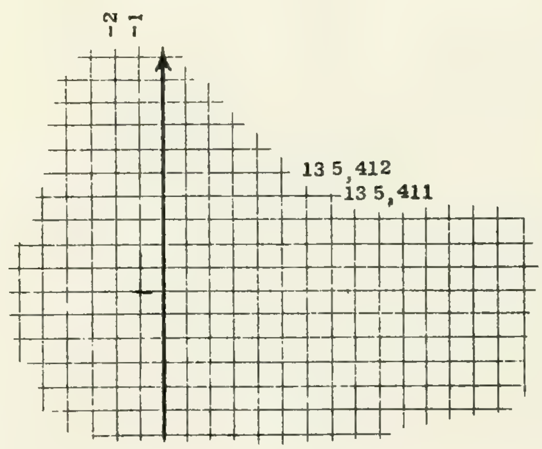

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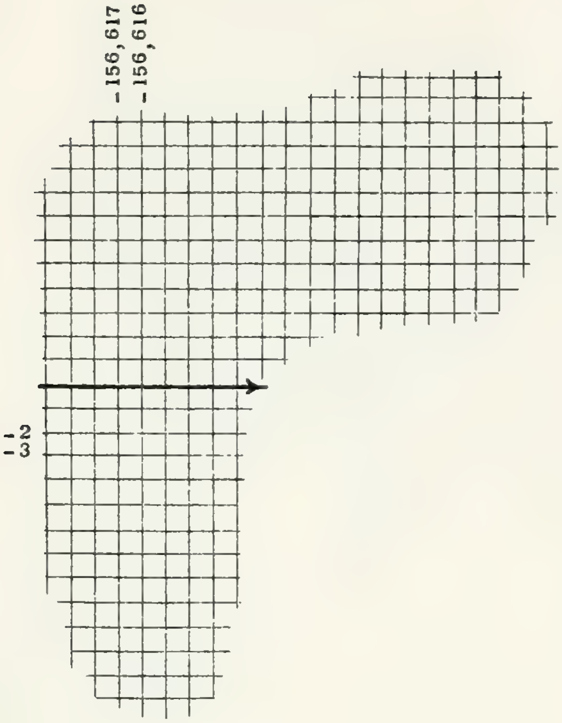

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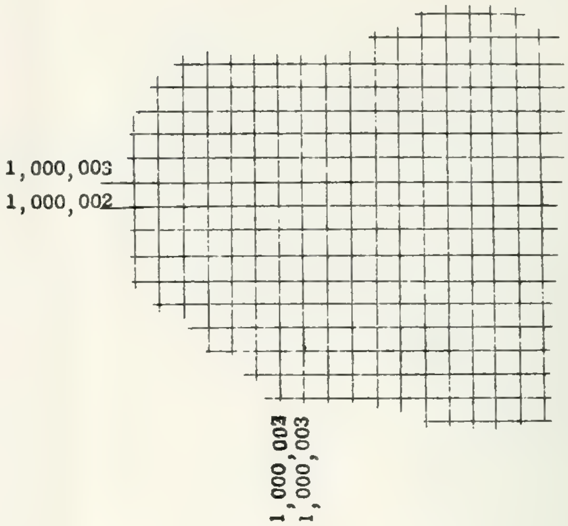

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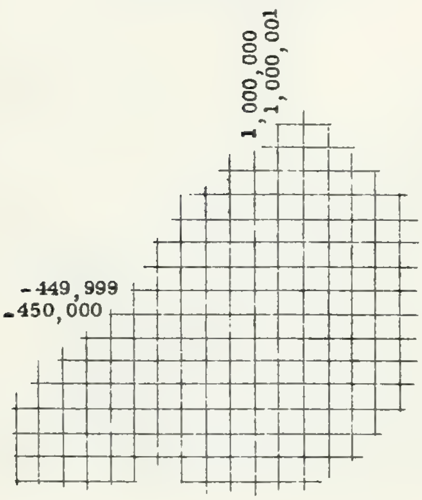

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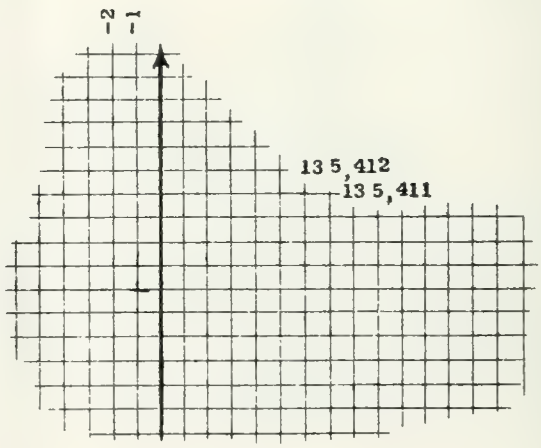

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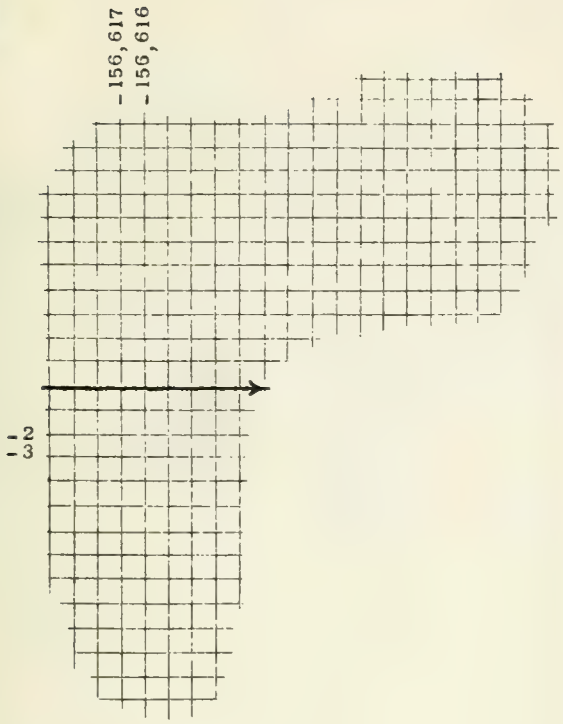

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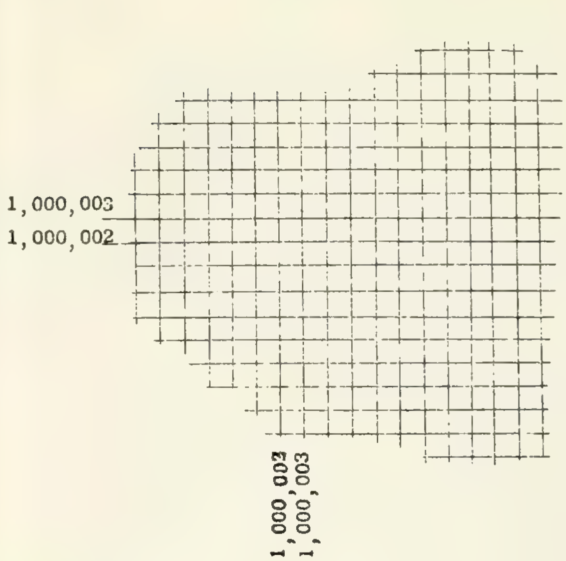

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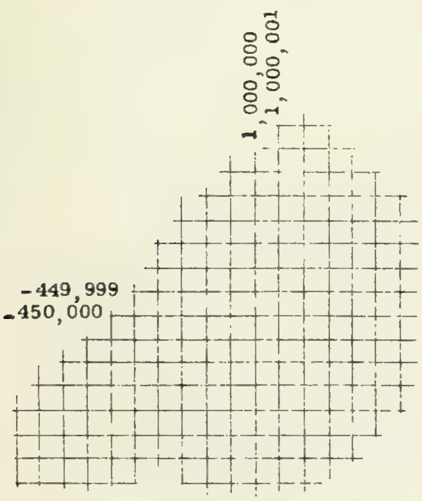

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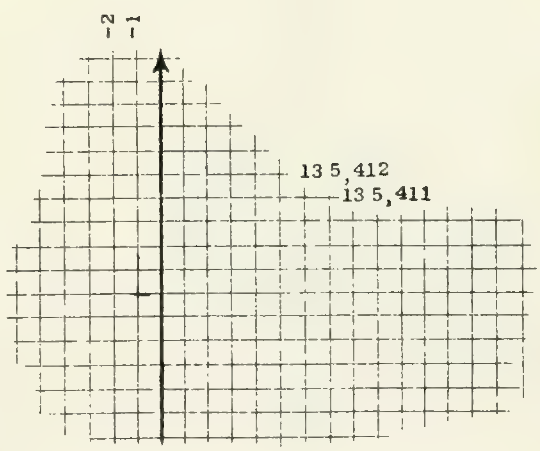

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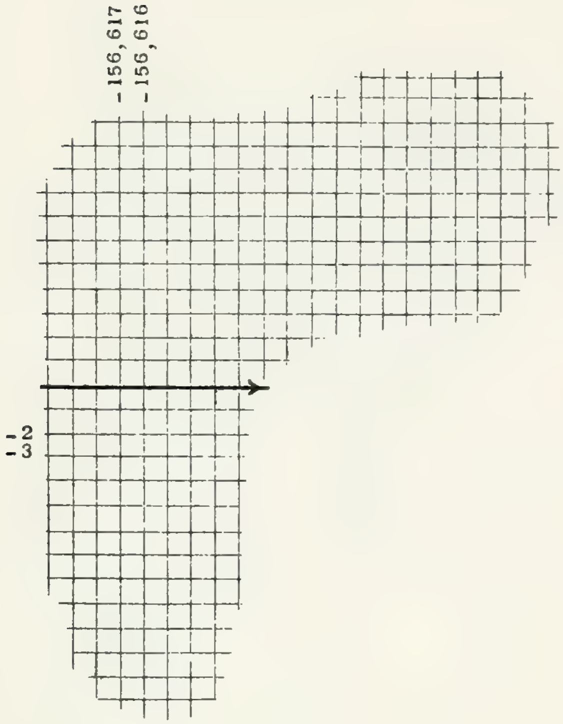

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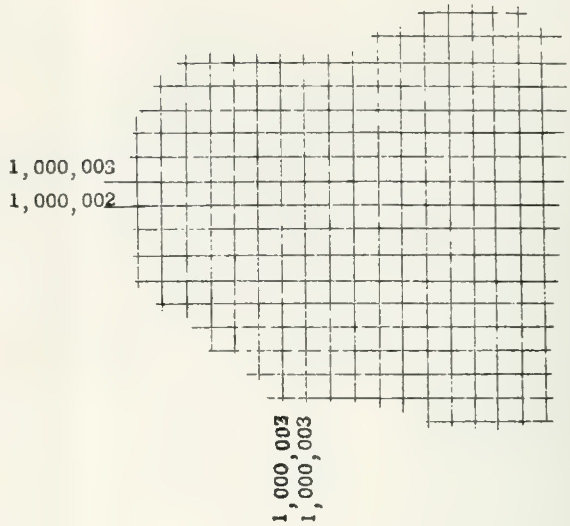

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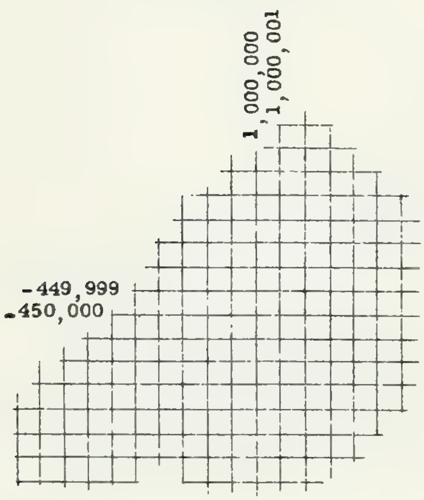

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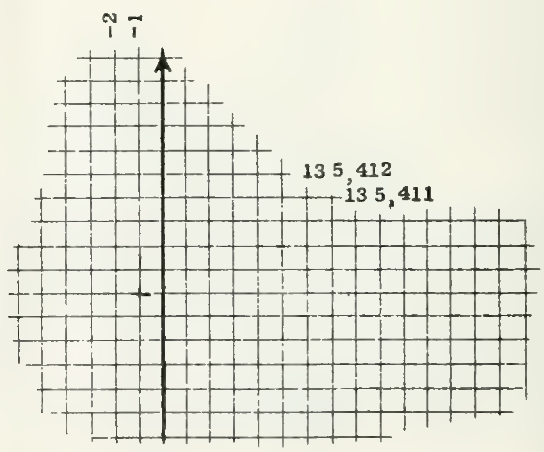

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